57. On Some Types of Polyhedra

By Keiô NAGAMI
Department of Mathematics, Ehime University
(Comm. by K. KUNUGI, M.J.A., April 12, 1956)

Let Q be a class of spaces having some topological property. According to O. Hanner [1], a space Y is called respectively an extensor for Q-spaces (an $ES(Q)$) or a neighborhood-extensor for Q-spaces (an $NES(Q)$) if every Y-valued mapping (=continuous transformation) defined on any closed subset C of any Q-space X always allows a continuous extension to the whole space X or to an open set G which contains C; and a space Y is called respectively an absolute retract for Q-spaces (an $AR(Q)$) or an absolute neighborhood retract for Q-spaces (an $ANR(Q)$) if Y is a Q-space and is a retract or a neighborhood retract of any Q-space containing Y as a closed subset. Analogously to these definitions we generalize the definition of an absolute n-retract which was given by C. Kuratowski [2] as follows: A space Y is called an n-$ES(Q)$ if every Y-valued mapping defined on any closed subset C of any Q-space X with an arbitrary small open set $G \supset C$ such that $\dim(X-G) \leqq n$, always allows a continuous extension to the whole space X; and a space Y is called an n-$AR(Q)$ if Y is a Q-space and is a retract of any Q-space or X containing Y as a closed subset where $\dim(X-G) \leqq n$ holds for an arbitrary small open set $G \supset C$. When Q is a class of metric spaces or of normal spaces, a Q-space Y which is an n-$ES(Q)$ is an n-$AR(Q)$ and conversely: This is essentially proved in [1, Theorem 8.1]. An n-sphere is a well-known example which is an n-AR (normal) [3, Theorem 6.1]. We shall study some types of polyhedra which are n-$ES(Q)$ for the case when Q is a class of metric spaces or of normal spaces.

Let $P=\{p_\alpha\}$ be an abstract set of points with $|P| \geqq n+1$, which will be called a vertex-set, where n is an arbitrary positive integer. The complex with the weak topology spanned by all m-simplexes, $m \leqq n$, whose vertices are mutually different points of P is called an *n-full-polyhedron* based on P and is denoted by $K(n, P)$. An n-sphere or an n-simplex is respectively nothing but an n-full-polyhedron based on P with $|P|=n+2$ or with $|P|=n+1$.

Theorem 1. *An n-full-polyhedron $K(n, P)$ is an n-ES (metric) for an arbitrary infinite vertex-set $P=\{p_\alpha; \alpha \in A\}$.*

Proof. Let X be a metric space, C be a closed subset of X with an arbitrary small open set D with $\dim(X-D) \leqq n$ and f be a mapping of C into $K(n, P)$. Let f_α be a non-negative real-valued function of C such that $f_\alpha(x)$ is a barycentric weight of $f(x)$ on p_α. Then f_α is continuous and $\{U_\alpha=\{x; f_\alpha(x)>0\}; \alpha \in A\}$ is an open covering of C whose order is at most $n+1$. First we shall show that there exists a locally finite open covering $\{V_\alpha; \alpha \in A\}$ of X whose order is at most $n+1$ such that $V_\alpha \frown C = U_\alpha$ for every $\alpha \in A$. Since any simplicial polyhedron with the weak topology is an *NES* (metric) [1, Theorem 25.1], f is continuously extended to g defined on some open set G with $G \supset C$. Let g_α be a non-negative real-valued function of G such that $g_\alpha(x)$ is a barycentric weight of $g(x)$ on p_α. Then $\{W_\alpha=\{x; g_\alpha(x)>0\}; \alpha \in A\}$ is an open covering of G whose order is at most $n+1$. Let $D \supset C$ be an open set of X with $\dim(X-D) \leqq n$. Let H and F be closed in X and E be open in X with $D \subset F \subset E \subset H \subset G$. Then an open covering $\{(W_\alpha - D) \smile (X-H); \alpha \in A\}$ of $X-D$ can be refined by a locally finite open covering $\{B_\alpha; \alpha \in A\}$ of $X-D$ whose order is at most $n+1$ such that $B_\alpha \subset (W_\alpha - D) \smile (X-H)$ for every $\alpha \in A$. Since $\{W_\alpha\}$ is locally finite in G [1, Lemma 25.4], $\{V_\alpha = (W_\alpha \frown E) \smile (B_\alpha - F); \alpha \in A\}$ is, as can easily be seen, a desired one.

Let h_α be a non-negative real-valued continuous function of X such that i) $h_\alpha | C = f_\alpha$, ii) $h_\alpha(x)=0$ if $x \in X-(C \smile V_\alpha)$, iii) $\{x; h_\alpha(x)>0\} = V_\alpha$. Let $h: X \rightarrow K(n, P)$ be a transformation such that $h(x)$ is the center of gravity of the vertices of $\{p_\alpha; x \in V_\alpha\}$ with the weights $h_\alpha(x)/\sum_{\beta \in A} h_\beta(x)$. Then h is continuous and satisfies $h | C = f$, which completes the proof.

Using the fact that every finite simplicial polyhedron is an *NES* (normal) [1, Theorem 27.4], we get the following theorem by the quite analogous method used in the above.

Theorem 2. *Every finite n-full-polyhedron is an n-ES (normal).*

Theorem 3. *A finite simplicial polyhedron L with* $\dim L \leqq n$ *is an n-ES (normal) if and only if L is a retract of a finite n-full-polyhedron.*

This is almost evident from the fact that L can be imbedded homeomorphically, as a subcomplex, into a finite n-full-polyhedron based on P which consists of suitable many vertices. On the other hand, the following analogous theorem is not trivial since a simplicial polyhedron with the weak topology is not always metrizable.

Theorem 4. *An infinite simplicial polyhedron L with the weak topology such that* $\dim L \leqq n$ *is an n-ES (metric) if and only if L is a retract of* $K(n, P(L))$, *where* $P(L)$ *denotes a set of all vertices of L.*

Proof. Since if-part is evident, we shall prove only-if-part. Suppose that L is an n-*ES* (metric) and consider L as a subcomplex of $K=K(n, P(L))$. Let $\{s_\lambda; \lambda \in \Lambda\}$ be a collection of all (closed) n-simplexes which are not contained in L and suppose that Λ is a well-ordered set which consists of all ordinals less than some fixed ordinal η. Since L is an n-ES (metric), there exists a retract-mapping $f_1: L_1 = L \smile s_1 \to L$. Let $L_\lambda = L \smile \{s_\xi; \xi \leqq \lambda\}$, μ be some fixed ordinal with $1<\mu<\eta$ and put the transfinite induction assumption that there exist retract-mappings $f_\nu: L_\nu \to L$, $\nu<\mu$, such that $f_\nu | L_\xi = f_\xi$ for any ν and ξ with $\xi<\nu$. Let us show the existence of a retract-mapping $f_\mu: L_\mu \to L$ such that $f_\mu | L_\nu = f_\nu$ for any $\nu<\mu$. Since $F = s_\mu \frown (L \smile \{s_\nu; \nu<\mu\})$ is a finite subpolyhedron of s_μ, we can choose a $\nu_0<\mu$ with $F = s_\mu \frown L_{\nu_0}$. Then $f_{\nu_0} | F$ has a continuous extension $g: s_\mu \to L$. Let $f_\mu: L_\mu \to L$ be a transformation such that i) $f_\mu | s_\mu = g$, ii) $f_\mu | s_\nu = f_\nu | s_\nu$ when $\nu<\mu$, iii) $f_\mu | L =$ the identity mapping. Then f_μ is continuous since the topology of K is the weak one. Moreover, it is evident that $f_\mu | L_\nu = f_\nu$ for every $\nu<\mu$, which completes the transfinite induction. Now we can construct for any $\lambda<\eta$ a retract-mapping $f_\lambda: L_\lambda \to L$ such that $f_\mu | L_\nu = f_\nu$ for any μ and ν with $\nu<\mu<\eta$. Let $f: K \to L$ be a transformation such that i) $f | s_\lambda = f_\lambda | s_\lambda$ for any $\lambda<\eta$, ii) $f | L =$ the identity mapping, and then f is continuous. Therefore L is a retract of K, which completes the proof.

References

[1] O. Hanner: Retraction and extension of mappings of metric and non-metric spaces, Ark. Mat., **2**, 315–360 (1952).
[2] C. Kuratowski: Sur les espaces localement connexes et péaniens en dimension n, Fund. Math., **24**, 269–287 (1935).
[3] K. Morita: On the dimension of normal spaces II, J. Math. Soc. Japan, **2**, 16–33 (1950).