76. On the σ-weak Topology of W^*-algebras

By Shôichirô SAKAI
Mathematical Institute, Tôhoku University, Sendai, Japan

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1. Preliminaries. The author [5] had shown that the σ-weak topology of W^*-algebras is free from the adjoint operation as follows:

Theorem A. *Suppose that a C^*-algebra M is the adjoint space of a Banach space F, then it is a W^*-algebra and the topology $\sigma(M, F)$ of M is the σ-weak topology.*

This will suggest that the following question is affirmative: *Suppose that ϕ is an algebraic isomorphism (not necessarily adjoint preserving) of a W^*-algebra onto another. Then, can we conclude that ϕ is σ-weakly bicontinuous?*

The purpose of this paper is to prove this in a more general form (§2, Theorem 2).

2. Theorems. Let M be a C^*-algebra, M^* the adjoint space of M.

Definition. A subspace V of M^* is said invariant, if $f \in V$ implies f_a, ${}_bf \in V$ for any $a, b \in M$, where $f_a(x)=f(xa)$ and ${}_bf(x)=f(bx)$.

Theorem 1. *Let V be an invariant subspace of M^* which is everywhere $\sigma(M^*, M)$-dense in M^*, then $V \cap S$ is everywhere $\sigma(M^*, M)$-dense in S, where S is the unit sphere of M^*.*

Proof. Put $T_af=f_a$ for $f \in V$, then T_a is a linear operator on the normed space V and moreover $\| T_af \| = \sup_{|||x|||\leq 1} |f(xa)| \leqq \| f \| \; |||a|||$; hence $\| T_a \| \leqq |||a|||$, where $\| T_a \|$ is the operator norm of T_a.

Suppose that $T_a=0$, then $(T_af)(x)=f(xa)=0$ for all $f \in V$ and $x \in M$. Since V is everywhere $\sigma(M^*, M)$-dense in M^*, $xa=0$ for all $x \in M$; hence $a=0$. Moreover $T_{ab}=T_aT_b$ and so the mapping $a \rightarrow T_a$ is an isomorphism; hence by the minimality of C^*-norm [cf. [1], Th. 10] $\| T_a \| = |||a|||$ for all $a \in M$. Therefore,

$$|||a||| = \sup_{|||x|||\leq 1,\, f\in V\cap S} |f(xa)| = \sup_{|||x|||\leq 1,\, f\in V\cap S} |{}_xf(a)|$$
$$\leqq \sup_{f\in V\cap S} |f(a)| \quad (\| {}_xf \| \leqq |||x||| \; \| f \|),$$

so that $|||a||| = \sup_{f\in V\cap S} |f(a)|$ for all $a \in M$; hence the bipolar of $V \cap S$ in E^* is S, that is, $V \cap S$ is everywhere $\sigma(M^*, M)$-dense in S. This completes the proof.

J. Dixmier [2] had shown a characterization of adjoint Banach spaces as follows: Let E be a Banach space, E^* the adjoint space of E and V be a subspace which is strongly closed and everywhere

$\sigma(E^*, E)$-dense in E^*. Then, if any strongly closed, proper subspace of V is not everywhere $\sigma(E^*, E)$-dense in E^*, V is said to be *minimal*.

Theorem B. *If there is a minimal subspace V in E^* such that $V \cap S$ is everywhere $\sigma(E^*, E)$-dense in S, where S is the unit sphere of E^*, then E is the adjoint space of V.*

Using Theorems A and B, we obtain immediately the following corollary of Theorem 1.

Corollary. *For a given C^*-algebra M, if there is an invariant minimal subspace F in its adjoint space, it is a W^*-algebra and $\sigma(M, F)$ is the σ-weak topology.*

It is noteworthy that the notion of positive functionals, which has played a principal rôle in the theory of W^*-algebras, is not employed in above discussions.

Now we shall deal with isomorphisms which are not necessarily adjoint preserving.

Let M be an AW^*-algebra in the sense of Kaplansky [4], N a C^*-algebra, S the unit sphere of M and ϕ be an isomorphism of M onto N.

Lemma 1. *Let (e_n) $(n=1, 2, \cdots)$ be a family of orthogonal projections of M and put $k_n = \sup_{x \geqq 0,\, x \in e_n M e_n \cap S} |||\phi(x)|||$, then there are an integer n_0 and a positive number k such that $k_n \leqq k$ for $n \geqq n_0$.*

Proof. Suppose that the lemma is false, then there is a sequence of positive elements (a_{nj}) $(a_{nj} \in e_{nj} M e_{nj} \cap S)$ such that $|||\phi(a_{nj})||| \geqq j$ $(j=1, 2, \cdots)$. On the other hand, let A be a maximal abelian *self-adjoint* subalgebra of M containing all a_{nj}, then $\phi(A)$ is a maximal abelian subalgebra of N, so that it is uniformly closed; hence by the uniqueness theorem of norm in semi-simple abelian Banach algebras [3; Satz 17], ϕ is bicontinuous on A. This contradicts the above inequality and completes the proof.

Lemma 2. *ϕ is uniformly continuous on M.*

Proof. Since M is a direct sum of a finite algebra and a purely infinite algebra, and ϕ becomes naturally adjoint preserving on the center of M, the proof is reduced to each case.

Case (1). Suppose that M is purely infinite, then the unit can be exhibited as the l.u.b. of $\aleph_0$ equivalent orthogonal projections (e_i) such that $e_i \sim I$. By Lemma 1 there is a projection e_{n_0} such that $\sup_{x \in e_{n_0} M e_{n_0} \cap S} |||\phi(x)||| \leqq 4k$. Let v be a partially isometric operator such that $v^*v = e_{n_0}$ and $vv^* = I$, then $x = v e_{n_0} v^* x v e_{n_0} v^*$ and if $|||x||| \leqq 1$, $|||e_{n_0} v^* x v e_{n_0}||| \leqq 1$, so that

$$\begin{aligned} |||\phi(x)||| &= |||\phi(v e_{n_0} v^* x v e_{n_0} v^*)||| \\ &= |||\phi(v)\, \phi(e_{n_0} v^* x v e_{n_0})\, \phi(v^*)||| \\ &\leqq 4\, |||\phi(v)||| \cdot |||\phi(v^*)||| \cdot k. \end{aligned}$$

This means that ϕ is continuous.

Case (2). Suppose that M is a finite algebra, then it is a direct sum of two algebras of types I and II. Therefore, the proof is reduced to each case.

(i) Now suppose that M is of type II, then there exists a sequence (e_j) of orthogonal projections such that for any e_j there is a projection f_j as follows: $e_jf_j=0$, $e_j\sim f_j$ and $e_{j-1}\sim e_j+f_j$ $(j\geqq 2)$, and $e_1f_1=0$, $e_1\sim f_1$ and $e_1+f_1=I$. Therefore, by Lemma 1 there is an integer j_0 such that $\sup_{x\in e_{j_0}Me_{j_0}\cap S} |||\phi(x)|||\leqq 4k$. On the other hand, there is a finite family $(p_i\,|\,i=1,2,\cdots,m)$ of orthogonal equivalent projections such that $p_1=e_{j_0}$ and $\sum_{i=1}^m p_i=I$.

Let v_i be a partially isometric operator such that $v_i{}^*v_i=p_i$ and $v_iv_i^*=e_{j_0}$, then $x=\left(\sum_{i=1}^m v_i^*e_{j_0}v_i\right)x\left(\sum_{i=1}^m v_i^*e_{j_0}v_i\right)=\sum_{i,j=1}^m v_i^*e_{j_0}v_ixv_j^*e_{j_0}v_j$. Since $|||x|||\leqq 1$ means $|||e_{j_0}v_ixv_j^*e_{j_0}|||\leqq 1$,

$$\begin{aligned}|||\phi(x)|||&\leqq\sum_{i,j=1}^m |||\phi(v_i^*)\phi(e_{j_0}v_ixv_j^*e_{j_0})\phi(v_j)|||\\&\leqq 4k\cdot\sum_{i,j=1}^m |||\phi(v_i^*)|||\cdot|||\phi(v_j)|||;\end{aligned}$$

hence ϕ is continuous.

(ii) Suppose that $M=\sum_{1\leqq i<\infty} Me_i$, where e_i is a central projection of M and Me_i is of type I_i.

By Lemma 1 there are an integer n_0 and a number k (>0) such that $k_n\leqq k$ for $n\geqq n_0$. Put $e=\bigvee_{n\geqq n_0} e_n$, then for any $x\in eMe\cap S$

$$\begin{aligned}|||\phi(exe)|||=|||\phi(e)\phi(x)\phi(e)|||&=\sup_{n\geqq n_0}|||\phi(e_n)\phi(x)\phi(e_n)|||\\&=\sup_{n\geqq n_0}|||\phi(e_nxe_n)|||\leqq 4k,\end{aligned}$$

for $(\phi(e_n))$ are orthogonal central projections of N, and $\phi(e)=\bigvee_{n\geqq n_0}\phi(e_n)$.

Moreover, for each Me_n $(\neq(0))$ there are orthogonal maximal abelian projections $(p_i\,|\,i=1,2,\cdots,n)$ such that $p_i\sim p_j$ and $\sum_{i=1}^n p_i=I$.

Since ϕ is continuous on the abelian algebra p_1Mp_1 from the proof of Lemma 1, we can conclude that ϕ is continuous on Me_n by the same method with (i), so that it is also continuous on $\sum_{i=1}^{n_0-1} Me_i$; finally it is so on M. This completes the proof.

Now we shall show the following theorem.

Theorem 2. *Let M be a W^*-algebra, N a C^*-algebra and ϕ be an isomorphism (not necessarily adjoint preserving) of M onto N, then N is a W^*-algebra and ϕ is σ-weakly bicontinuous.*

Proof. By Lemma 2, ϕ is uniformly continuous, so that it is bicontinuous by the classical theorem of Banach space. Let M^* and

N^* be the adjoint spaces of M and N respectively, and M_* be the subspace of M^* composed of all σ-weakly continuous linear functionals on M.

Then, for any $y^* \in M^*$ and $x \in N$

$$(\phi^{-1}(x), y^*)_M = (x, \tilde{\phi}^{-1}(y^*))_N,$$

where $(a, b)_M$(resp. $(a', b')_N$) is the value at a (resp. a') of a linear functional b of M^* (resp. b' of N^*), and $\tilde{\phi}^{-1}$ is the adjoint mapping of ϕ^{-1}.

Since ϕ^{-1} is uniformly bicontinuous, $\tilde{\phi}^{-1}$ is a bicontinuous mapping of M^* with the topology $\sigma(M^*, M)$ onto N^* with the topology $\sigma(N^*, N)$, and so $\tilde{\phi}^{-1}(M_*)$ is a minimal subspace of N^*. Moreover, if $\tilde{\phi}^{-1}(\eta) \in \tilde{\phi}^{-1}(M_*)$, then

$$\begin{aligned} {}_b\tilde{\phi}^{-1}(\eta)_a(x) &= \tilde{\phi}^{-1}(\eta)(bxa) = (bxa, \tilde{\phi}^{-1}(\eta))_N \\ &= (\phi^{-1}(bxa), \eta)_M = (\phi^{-1}(b)\phi^{-1}(x)\phi^{-1}(a), \eta)_M \\ &= (\phi^{-1}(x), {}_{\phi^{-1}(b)}\eta_{\phi^{-1}(a)})_M = (x, \tilde{\phi}^{-1}({}_{\phi^{-1}(b)}\eta_{\phi^{-1}(a)}))_N. \end{aligned}$$

Since ${}_{\phi^{-1}(b)}\eta_{\phi^{-1}(a)}$ belongs to M_*, ${}_b\tilde{\phi}^{-1}(\eta)_a$ belongs to $\tilde{\phi}^{-1}(M_*)$, so that $\tilde{\phi}^{-1}(M_*)$ is an invariant minimal subspace of N^*; hence by the corollary of Theorem 1, N is a W^*-algebra. As $\sigma(N, \tilde{\phi}^{-1}(M_*))$ is the σ-weak topology, ϕ is σ-weakly bicontinuous.
This completes the proof.

Remark 1. In this paper, the existence of the unit in C^*-algebras is not assumed, and particularly Theorem A implies that an adjoint C^*-algebra has necessarily the unit. More generally we can show the following theorem [cf. [5]; Appendix]: *A C^*-algebra has the unit if and only if its unit sphere has an extreme point.*

Remark 2. Lemma 2 is immediately obtained from the result of C. E. Rickart [Ann. Math., **51**, 615-628 (1950)] which assures that an isomorphism of a C^*-algebra onto another is uniformly continuous.

But, as our proof is extensible to some *into*-isomorphisms of AW^*-algebras, it is stated.

References

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