94. Some Classes of Riemann Surfaces Characterized by the Extremal Length

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In this article we shall consider some classes of Riemann surfaces characterized by the extremal length and state their properties, the detailed proofs of which will be given in another paper[1] together with other related results.

1. Let $\{c\}$ $(\neq\phi)$ be a system of curves each of which consists of a finite or countable number of curves on an arbitrary Riemann surface R. For any non-negative covariant ρ on R such that

$$\underline{\int_c} \rho(z)\,|dz|\geqq 1, \text{ for all } c\in\{c\},$$

the extremal length $\lambda\{c\}$ with respect to $\{c\}$ is defined by

$$\lambda\{c\}^{-1}=\inf_\rho \overline{\iint_R} \rho^2(z)\; dx\,dy, \text{ where } z=x+iy \text{ is a uniformizer.}$$

Now we consider the system of curves $\{C\}\subset R-R_0$ (R_0 is an image of z-circle) such that each $C\in\{C\}$ consists of a finite number of disjoint *analytic* Jordan closed curves and C is homologous to ∂R_0 (the boundary of R_0). Then we can prove

PROPOSITION 1. *R is of parabolic type if and only if $\lambda\{C\}=0$.*

2. Let $\{\gamma\}$ be a subset of $\{C\}$ which contains an infinite number of curves of $\{C\}$ tending to the ideal boundary $\mathfrak{J}$ of R. Then we can prove the property which plays a fundamental role in the following.

PROPOSITION 2. *Suppose that φ_1 and φ_2 are any two non-negative covariants which are square integrable over $R-K$ (K is a compact domain with analytic boundaries). If $\lambda\{\gamma\}=0$, then there exists a sequence of curves $\gamma_n\in\{\gamma\}$ $(\gamma_n\cap K=\phi)$ tending to $\mathfrak{J}$ such that*

$$\underline{\int_{\gamma_n}}\varphi_1\,|dz|\underline{\int_{\gamma_n}}\varphi_2\,|dz|\to 0 \quad \textit{for } n\to\infty.$$

3. We take account of two subsets $\{\Gamma\}$, $\{L\}_E$ of $\{C\}$ as $\{\gamma\}$.

(I) $\{\Gamma\}$: $\{\Gamma\}$ denotes the set of curves $\Gamma\in\{C\}$ such that in the decomposition of Γ into its components each component divides R into two disjoint parts.

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1) Kusunoki, Y.,: On Riemann's periods relations on open Riemann surfaces, Mem. Coll. Sci., Univ. Kyoto, Ser. A, Math., **30**, No. 1 (shortly appear).

(II) $\{L\}_E$: This is defined for an exhaustion $E=\{R_n\}$ such that $\partial R_n \equiv L_n \in \{\Gamma\}$. That is, $\{L\}_E=\bigcup_{n=1}^{\infty}\{L_n\}$, where $\{L_n\}$ is the set of curves of $\{\Gamma\}$ contained in annuli[2] including L_n.

First of all we note that $\{\Gamma\}$ and $\{L\}_E$ contain an infinite number of curves tending to $\mathfrak{J}$.[3] We shall denote by O' or O'' the classes of Riemann surfaces for which $\lambda\{\Gamma\}=0$ respectively $\lambda\{L\}_E=0$ for a certain exhaustion E. Since $\{L\}_E \subset \{\Gamma\} \subset \{C\}$ and $\lambda\{C\}=0$ characterizes the class O_G (Prop. 1), we have $O'' \subset O' \subset O_G$.

THEOREM 1. *If $R \in O'$ and $u(p)$ is a single-valued bounded harmonic function on $R-K$, then $u(p)$ has always a limit when p tends to any ideal boundary element of $\mathfrak{J}$.*

This theorem can be proved by using Prop. 2, the maximum and minimum principle on $R-K$ and Nevanlinna's theorem which states: u has a finite Dirichlet integral over $R-K$ if and only if u is bounded.

Next we shall show a sufficient condition for which R should belong to the class O'' therefore to O'. Let D_n, $n=1, 2, \cdots$, be a *disjoint* sequence of annuli including the curves $L_n \in \{\Gamma\}$ and $\{c_n\}$ be the set of curves of $\{\Gamma\}$ lying in D_n, then it is proved that

$$\lambda\{c_n\}=2\pi/\log \mu_n$$

where μ_n denotes the Sario-Pfluger's ring modul of D_n. Since $D_m \cap D_n=\phi$ $(n \neq m)$,

$$\lambda\{\Gamma\}^{-1} \geqq \lambda\{L\}_E^{-1} \geqq \lambda\{\bigcup_{n=1}^{N}\{c_n\}\}^{-1}=\sum_{n=1}^{N}\lambda\{c_n\}^{-1},$$

hence we have the following

THEOREM 2. *Let D_n, $n=1, 2, \cdots$ be a disjoint sequence of annuli including the curves of $\{\Gamma\}$ and let μ_n denote the modul of D_n. If $\prod_{n=1}^{\infty}\mu_n$ diverges, then we have $R \in O'' \subset O'$.*

By Theorem 2 we can see that Theorem 1 gives a sharp generalization of Heins' sufficient condition.[4] Using Theorem 2 we can also prove that $O''=O'=O_G$, if R is of finite genus, and that $O'' \subset O' \subsetneqq O_G$ in general, since there exists an example of parabolic Riemann surface R such that single-valued bounded harmonic functions defined on $R-K$ do not have a limit at the ideal boundary.[5]

4. Let $A_1, B_1, \cdots, A_n, B_n, \cdots$ denote a canonical homology basis on an arbitrary Riemann surface R such that for an exhaustion $\{R_n\}$

2) By annulus including $l \in \{\Gamma\}$ we mean the union of doubly connected ring domains each of which includes a component of l.

3) Sario, L.,: An extremal method on arbitrary Riemann surfaces, Trans. Amer. Math. Soc., **73**, 466 (1952).

4) Heins, M.,: Riemann surfaces of infinite genus, Ann. Math., **55** (1952).

5) Heins, M.,: Loc. cit.

of R, $A_1, B_1, \cdots, A_{k_n}, B_{k_n}$ are the relative homology basis mod ∂R_n.[6] We shall call such basis a canonical homology basis of $\mathfrak{A}$-*type* with respect to $\{R_n\}$. Now let $R \in O'$, then we take the exhaustion $\{R^\nu\}$ of R such that $\partial R^\nu = \gamma_\nu$ where γ_ν are the curves of $\{\Gamma\}$ defined by Prop. 2. Let $A_1, B_1, \cdots, A_n, B_n, \cdots$ be a canonical homology basis of $\mathfrak{A}$-type with respect to $\{R^\nu\}$. Let df_j $(j=1, 2)$ be any two Abelian differentials of the first kind with finite Dirichlet integrals over R. Since $\gamma_\nu \in \{\Gamma\}$, it follows immediately that for the fixed branch[7] of f_1

$$\left|\int_{\gamma_\nu} f_1\, df_2\right| \to 0 \quad \text{for} \quad \nu \to \infty .$$

Therefore we can prove the following

THEOREM 3. *For each Riemann surface $R \in O'$, there exist an exhaustion and the corresponding canonical homology basis of $\mathfrak{A}$-type such that for any two Abelian differentials $df_j = du_j + idv_j$ $(j=1, 2)$ with finite Dirichlet integrals over R, we have*

$$\lim_{\nu\to\infty} \sum_{i=1}^{k_\nu} \left(\int_{A_i} df_1 \int_{B_i} df_2 - \int_{B_i} df_1 \int_{A_i} df_2 \right) = 0,$$

$$\iint_R \operatorname{grad} u_1 \operatorname{grad} u_2\, dx\, dy = \lim_{\nu\to\infty} \sum_{i=1}^{k_\nu} \left(\int_{A_i} du_1 \int_{B_i} dv_2 - \int_{B_i} du_1 \int_{A_i} dv_2 \right).$$

Especially when $R \in O''$, i.e. $\lambda\{L\}_E = 0$, these Riemann's relations hold always for the canonical basis of $\mathfrak{A}$-type with respect to the exhaustion E.[8]

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6) Ahlfors, L.,: Normalintegrale auf offenen Riemannschen Flächen, Ann. Acad. Sci. Fenn., Ser. A, **35** (1947).

7) Cf. Ahlfors, L.,: Loc. cit.

8) Cf. Pfluger, A.,: Über die Riemannsche Periodenrelation auf transzendenten hyperelliptischen Flächen, Comm. Math. Helv., **30** (1956).