92. *On the Cells of Symplectic Groups*

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1. Among the cellular decomposition problems of the classical Lie groups (the special orthogonal group $SO(n)$, the special unitary group $SU(n)$, and the symplectic group $Sp(n)$), a cellular decomposition of $SO(n)$ was given by J. H. C. Whitehead[1] and recently that of $SU(n)$ was given by the author.[2] In this paper, we shall give a cellular decomposition of $Sp(n)$. The details will appear in the Journal of the Institute of Polytechnics, Osaka City University.

2. Let Q^n be a vector space of dimension n over the field of quaternion numbers, and e_i be the element of Q^n whose i-th coordinate is 1 and whose other coordinates are 0. We embed Q^{n-1} in Q^n as a subspace whose first coordinate is 0. Let S^{4n-1} be the unit sphere in Q^n.

Let $Sp(n)$ be the group of all symplectic linear transformations of Q^n. Put $\pi(A)=Ae_1$ for $A\in Sp(n)$. Then we have a fibre space $Sp(n)/Sp(n-1)=S^{4n-1}$ with projection $\pi: Sp(n)\to S^{4n-1}$.

3. Let E^{4n-4} be a closed cell consisting of all $x=(x_2, x_3, \cdots, x_n)$, where $x_2, x_3, \cdots, x_n$ are quaternion numbers such that $|x_2|^2+|x_3|^2+\cdots+|x_n|^2=1$, and let E^3 be a closed cell consisting of all pure imaginary quaternion numbers whose norms are $\leqq 1$.

Now, we shall define a map $f: E^{4n-1}=E^{4n-4}\times E^3\to Sp(n)$ by

$$f(x, q)=(\delta_{ij}+x_i p\bar{x}_j),\quad i, j=1, 2, \cdots, n,$$

where $x_1=\sqrt{1-(|x_1|^2+|x_2|^2+\cdots+|x_n|^2)}$ and $p=2\sqrt{1-|q|^2}(q-\sqrt{1-|q|^2})$. It will be easily verified that $f(x, q)$ is symplectic.

4. Define a map $\xi: E^{4n-1}\to S^{4n-1}$ by $\xi=\pi f$, then we have the

Lemma. *ξ maps $\mathcal{E}^{4n-1}=E^{4n-1}-(E^{4n-1})^{\bullet}$ homeomorphically onto $S^{4n-1}-e_1$ and maps $(E^{4n-1})^{\bullet}$ to a point e_1.*

From this lemma, we can see that f maps $\mathcal{E}^{4k-1}$ homeomorphically into $Sp(k)\subset Sp(n)$ for $n\geqq k\geqq 1$.

5. For $n\geqq k_1>k_2>\cdots>k_j\geqq 1$, extend f to a map $f: E^{4k_1-1}\times E^{4k_2-1}\times\cdots\times E^{4k_j-1}\to Sp(n)$ by

$$\bar{f}(y_1, y_2, \cdots, y_j)=f(y_1)f(y_2)\cdots f(y_j).$$

1) J. H. C. Whitehead: On the groups $\pi_r(V_{n,m})$ and sphere bundles, Proc. London Math. Soc., **48** (1945).

2) I. Yokota: On the cell structures of $SU(n)$ and $Sp(n)$, Proc. Japan Acad., **31** (1955). The results given therein are incorrect for $Sp(n)$. The present paper is a correction for the part of $Sp(n)$.

Put $\begin{cases} e^{4k_1-1,4k_2-1,\cdots,4k_j-1}=\bar{f}(\mathcal{E}^{4k_1-1}\times\mathcal{E}^{4k_2-1}\times\cdots\times\mathcal{E}^{4k_j-1}), \\ e^0=I_n.^{3)} \end{cases}$

Then we have the following results.

Theorem 1. *The symplectic group $Sp(n)$ is a cell complex composed of 2^n cells e^1 and $e^{4k_1-1,4k_2-1,\cdots,4k_j-1}$ with $n\geqq k_1>k_2>\cdots>k_j\geqq 1$. The dimension of $e^{4k_1-1,4k_2-1,\cdots,4k_j-1}$ is* $(4k_1-1)+(4k_2-1)+\cdots+(4k_j-1)$.

For such a cell structure of $Sp(n)$ given in this theorem, the boundary homomorphisms are trivial in all dimensions.

Theorem 2. *$Sp(n)$ has no torsion groups, and its Poincaré polynomial is*

$$P_{Sp(n)}(t)=(1+t^3)(1+t^7)\cdots(1+t^{4n-1}).$$

6. Remark. *The 7-dimensional cell e^7 is attached to the 3-dimensional cell e^3 by the Blaker-Massey's map ν (i.e. $\nu: S^6\to S^3$* is obtained by applying Hopf construction to a map $\rho: S^3\times S^2=(E^4)^{\bullet}\times(E^3)^{\bullet}\to S^2=(E^3)^{\bullet}$ such that $\rho(p,q)=pq\bar{p}$).

3) I_n is the identity linear transformation of Q^n.