90. Notes on Topological Spaces. IV. Function Semiring on Topological Spaces

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(Comm. by K. KUNUGI, M.J.A., June 12, 1956)

In our previous paper [3], we generalized some results on the theory of the space of maximal ideals of a semiring by W. Slowikowski and W. Zawadowski [4]. In this paper, we shall consider the relation between a function semiring on a normal space S and a lattice of closed sets in S. By using the result of it, we shall prove some theorems on function semiring. Such a consideration for function ring was also treated by G. Higman [2].

Let S be a T_2-space. Let $C^+(S)$ be the set of all continuous, bounded, non-negative real-valued functions on S, and let L be the lattice of all closed sets in S. $C^+(S)$ is a semiring*) with respect to the usual addition and multiplication and further $C^+(S)$ is a *positive semiring* in the sense of W. Slowikowski and W. Zawadowski [4].

We assume that we are familiar with the notions of proper ideals, maximal ideals of $C^+(S)$ and proper filter, ultrafilter of L (see, K. Iséki and Y. Miyanaga [3], and N. Bourbaki [1]). Following G. Higman [2], we shall first give a correspondence between ideals of $C^+(S)$ and filters of L.

Let I be an ideal of $C^+(S)$, then we shall define a set I^* of closed sets of S as follows: $A \in I^*$ if and only if, for any closed set F not meeting A, there is a function f of I such that the lower bound of f on F is positive, i.e. $\inf f(x) > 0$ on F.

Let J be a filter of L, and $f \in J^*$ if and only if, for every positive ε, there is a closed set A of J such that $f < \varepsilon$ on A.

Let I be a proper ideal of $C^+(S)$, and $A \in I^*$, then it is clear that $A \cup B \in I^*$ for every $B \in L$. Let A, $B \in I^*$, and let F be a closed set such that $F \frown (A \frown B) = 0$. Then $F \frown B$ does not meet A, therefore there is a function $f_1 \in I$ such that the lower bound α_1 of f_1 on $F \frown B$ is positive. For $\frac{1}{2}\alpha_1$, let $F_1 = \left\{x \mid f_1(x) \leq \frac{1}{2}\alpha_1\right\}$, then F is non-empty closed set and $F \frown F_1$ does not meet B. Therefore there is a function f_2 of L such that $\inf f_2 = \alpha_2$ is positive on $F \frown F_1$. Since I is an ideal, $f_1 + f_2 \in I$ and $f_1(x) + f_2(x) \geq \min\left(\frac{1}{2}\alpha_1, \alpha_2\right)$ on F. Hence I^* is a filter.

*) For the precise definition of semirings, see K. Iséki and Y. Miyanaga [3].

We shall show $(O)^*=S$. Suppose $(O)^* \ni A \neq S$, then, for a point x such that $x \in S-A$, $O(p)=0$ and hence $A \notin (O)^*$. By the definition of *-operation, $(O)^* \supset (S)$. Hence $(O)^*=S$.

Suppose that $I \neq (O)$, then I contains a function f such that $f(x)=\alpha>0$ for some point x. The set $F=\left\{x \mid f(x) \leq \frac{\alpha}{2}\right\}$ is closed and $F \neq S$, and $F \in I^*$. This implies $I^* \neq (S)$. Therefore, if $I^*=(S)$, then $I=(O)$. Clearly $(C^+(S))^*=L$. On the other hand, let $I^*=L$, then I^* contains the empty set. By the definition of *-operation, I contains a function f such that $\inf f>0$ on S. Hence $C^+(S) \ni f^{-1}$ and $ff^{-1}=1$ is contained in I. Therefore $I=C^+(S)$. This implies the following

Proposition 1. I^* is a filter of L. I^* is a proper filter if and only if I is a proper ideal.

Now we shall prove the similar proposition for ideals in $C^+(S)$.

Let J be a filter of L, and let f, $g \in J$, then, for any positive ε, there are two closed sets A, B of J such that $f(x)<\frac{1}{2}\varepsilon$ on A and $g(x)<\frac{1}{2}\varepsilon$ on B. Since J is a filter, $A \frown B \in J$ and $f(x)+g(x)<\varepsilon$ on $A \frown B$. Therefore $f+g \in J^0$. Let $f \in J^0$ and $g \in C^+(S)$, $g<\delta$, then, for any positive ε, there is a closed set A such that $f(x)<\frac{\varepsilon}{\delta}$ on A. Hence $f(x)\, g(x)<J^0$ on A. Therefore $fg \in J^0$. This shows that J^0 is an ideal.

To prove that J^0 is a proper ideal, if and only if J is a proper filter, we shall assume S is a completely regular space. We shall show the following four relations.

1) $(S)^0=(O)$.

Let $f \in (S)^0$, then, for any positive ε, $f(x)<\varepsilon$ on S. Hence $f=0$.

2) $J^0=(O)$ implies $J=(S)$.

Suppose that $J \ni A \neq S$, then, by the completely regularity of S, there is a non-zero function on S such that $f(x)=0$ on A. Therefore $f \in J^0$ and we have $J^0 \neq (O)$.

3) $L^0=C^+(S)$ is clear.

4) $J^0=C^+(S)$ implies $J=L$.

Since J^0 contains the unit function $f(x)=1$, J contains the empty set. Hence $J=L$.

Proposition 2. J^0 is an ideal of $C^+(S)$. If S is completely regular, J^0 is a proper ideal of $C^+(S)$, if and only if J is a filter of L.

Proposition 3. For ideals I_λ, $(\bigcap_\lambda I_\lambda)^* \subset \bigcap_\lambda I_\lambda^*$. For filter J_λ, $(\bigcap_\lambda J_\lambda)^0 \subset \bigcap_\lambda J_\lambda^0$.

If, for two ideals I_1 and I_2, $I_1 \subset I_2$, then $I_1^* \subset I_2^*$.

If, for two filters J_1 and J_2, $J_1 \subset J_2$, then $J_1^0 \subset J_2^0$.

Proof. Let $A \in (\bigcap_\lambda I_\lambda)^*$, and let F be a closed set such that $A \frown F = 0$, then there is a function f of $\bigcap_\lambda I_\lambda$ such that inf f on F is positive. For every λ, $f \in I_\lambda$, and hence $F \in I_\lambda^*$. Therefore $A \in \bigcap_\lambda I_\lambda^*$. Similarly we have $(\bigcap_\lambda J_\lambda)^0 \subset \bigcap_\lambda J_\lambda^0$. If $I_1 \subset I_2$, then $I_1 = I_1 \frown I_2$. Hence

$$I_1^* = (I_1 \frown I_2)^* \subset I_1^* \frown I_2^*.$$

This implies $I_1^* \subset I_2^*$.

Proposition 4. $(I^*)^0 \supset I$ for every ideal I of $C^+(S)$.

Proof. Let $f \in I$ and let ε be a positive number. Then $A = \left\{x \mid f(x) \leq \frac{1}{2}\varepsilon\right\}$ is a closed set. For every closed set not meeting A, $f(x) \geq \frac{1}{2}$ on it, and hence $A \in I^*$. From $f(x) < \varepsilon$ on A, $f \in (I^*)^0$. Therefore $I \subset (I^*)^0$.

Proposition 5. $(J^0)^* \supset J$ for every filter J of L for a normal space S.

Proof. Let A be a closed set, and let F be a closed set such that $A \frown F = 0$. By the normality of S there is a function $f \in C^+(S)$, such that $f(x) = 0$ on A and $f(x) = 1$ on F. From $f(x) = 0$ on A and $A \in J$, $f \in J^0$. Therefore, there is a function of J such that it admits a positive lower bound on F. This implies $A \in (J^0)^*$. Hence $J \subset (J^0)^*$.

An ideal I of $C^+(S)$ is called *closed* if $(I^*)^0 = I$.

A filter J of L is called *closed* if $(J^0)^* = J$. By Propositions 1 and 2, (O), $C^+(S)$ are closed ideals (S). L is a closed filter for a completely regular space. Let M be a maximal ideal of $C^+(S)$ for a completely regular space, then $(M^*)^0 \supset M$. Hence $(M^*)^0 = M$ or $C^+(S)$. If $(M^*)^0 = C^+(S)$, then $M^* = L$ and $M = C^+(S)$. Hence M is not maximal. Therefore $(M^*) = M$ and M is a closed ideal. If S is normal, any maximal filter of L is closed.

Proposition 6. For a completely regular space S, (O), $C^+(S)$ are closed ideals, and (S), L are closed filters. Any maximal ideal of $C^+(S)$ is closed. For a normal space, every ultrafilter of L is closed.

Let I_λ be closed ideals of $C^+(S)$ for every λ. Then

$$\{(\bigcap_\lambda I_\lambda)^*\}^0 \subset (\bigcap_\lambda I_\lambda^*)^0 \subset \bigcap_\lambda (I_\lambda^*)^0 = \bigcap_\lambda I_\lambda.$$

Hence $\bigcap_\lambda I_\lambda$ is a closed ideal. Let J be a filter of L for normal space, then $\bigcap_\lambda I_\lambda$ is closed.

Let I be a closed ideal of $C^+(S)$, then $I = (I^*)^0$ and hence $I^* = \{(I^*)^0\}^*$, therefore I^* is a closed filter of L. Let J be a closed filter of L, then J^0 is a closed ideal in $C^+(S)$. Hence, if I and J are closed ideal and closed filter respectively, and $I^* = J$, then $I = J^0$. If $J^0 = I$, then $I^* = J$. This implies that there is one-to-one correspondence between the set of closed ideals and closed filters.

If I is a maximal ideal for a completely regular space, then I is a closed ideal. We shall show that I^* is an ultrafilter. To prove it, suppose that I^* is not ultrafilter, then there is proper ideal J such that $I^* \subset J \subset L$ and $I^* \neq J$. We have $I = (I^*)^0 \subset J^0 \subset C^+(S)$. Since the correspondence $J \to J^0$ is one-to-one, I is contained in J^0 properly, hence I is not maximal. Therefore, I^* is an ultrafilter. By the same method, if J is an ultrafilter for a normal space, J^0 is a maximal ideal.

Proposition 7. There is one-to-one correspondence between the set of closed ideals and the set of closed filters. For a completely regular space, by the correspondence, every maximal ideal goes to an ultrafilter. For a normal space, every ultrafilter corresponds to a maximal ideal.

Suppose that S is a completely regular space.

Let $I(a)$ be the set $\{f \mid f(a)=0,\ f \in C^+(S)\}$, and let F be a closed set such that $F \not\ni a$, then there is a function $f \in C^+(S)$ such that $f(a)=0$ and $f(x)=1$ on F. Such a function f is obtained in $I(a)$. By the definition of $(I(a))^*$, every closed set F containing a is in $(I(a))^*$. On the other hand, let A be a closed set not meeting a, it is clear that $A \notin (I(a))^*$. Therefore, the ideal $I(a)$ corresponds to the filter of closed set containing a. By Proposition 7, the set of all closed sets containing a given point a for a normal space S is ultrafilter and hence the ideal $I(a)$ is maximal in $C^+(S)$.

Proposition 8. By the correspondence of Proposition 7, for a completely regular space, every ideal $I(a)$ corresponds to the filter of closed sets containing a.

Proposition 9. Any ideal $I(a)$ for a normal space is maximal.

Let α be the operation $*0$, then

(1) $I^\alpha \supset I$.

(2) $I^{\alpha\alpha} - I^\alpha$.

(3) $I_1 \supset I_2$ implies $I_1^\alpha \supset I_2^\alpha$.

(4) $O^\alpha = 0$.

Therefore $I \to I^\alpha$ is a closure operation in $C^+(S)$. If $I_1 \frown I_2 = 0$, then $I_1^\alpha \frown I_2^\alpha = 0$. Suppose that $I_1^\alpha \frown I_2^\alpha \ni f$ and $f \neq 0$, then we can find $A \in I_1^*$, $B \in I_2^*$ such that $A \neq S \neq B$ and $A \smile B \neq S$. Hence $A \smile B \in I_1^* \frown I_2^*$. Therefore, for a non-empty closed set F such that $(A \smile B) \frown F = 0$, there are $g_1 \in I_1$, $g_2 \in I_2$ and $g_1(x) \geq \varepsilon_1$, $g_2(x) \geq \varepsilon_2$ on F. This shows that $g_1 g_2$ is non-zero function and $g_1 g_2 \in I_1 \frown I_2$, which is a contradiction. Therefore

(5) $I_1 \frown I_2 = 0$ implies $I_1^\alpha \frown I_2^\alpha = 0$.

References

[1] N. Bourbaki: Topologie Générale, Chaps. I–X, Hermann, Paris.

[2] G. Higman: The compacting of topological space, Quarterly Jour. Math., **17**, 27–32 (1948).

[3] K. Iséki and Y. Miyanaga: Notes on topological spaces. III. On space of maximal ideals of semiring, Proc. Japan Acad., **32**, 325–328 (1956).

[4] W. Slowikowski and W. Zawadowski: A generalisation of maximal ideals method of Stone-Gelfand, Fund. Math., **42**, 216–231 (1955).