102. *Note on Algebras of Bounded Representation Type*

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1. Let A be an associative algebra with a unit element over a field k, N be the radical of A and $g_A(d)$ be the number of inequivalent indecomposable representations of A of degree d where d is an integer. Now A is said to be of *bounded representation type* if there exists an integer d_0 such that $g_A(d)=0$ for all $d\geqq d_0$ and A is said to be of *finite representation type* if $\sum_d g_A(d)$ is finite.[1]

Concerning these classes of algebras of bounded type and finite type, Professor Brauer and Professor Thrall conjectured that these two classes are identical.[2] This conjecture is not yet proved, but now we shall prove it in a special case where $N^2=0$ and k is algebraically closed.

2. By the same way as [2] we may assume that A is the basic algebra. Then for this purpose we have only to prove that arbitrary two representations by indecomposable A-left modules $\mathfrak{M}_1=\sum_{i=1}^{r}\sum_{\lambda_i=1}^{s_i}Ae_im_{i,\lambda_i}$ and $\mathfrak{M}_2=\sum_{i=1}^{r}\sum_{\lambda_i=1}^{s_i}Ae_im'_{i,\lambda_i}$ which have the same type are equivalent. For the number of such A-left modules with different types is finite.[3]

Now from the proof of the main theorem of [2], we may consider about the following three cases:

(a) $\{Ne_1,\cdots,Ne_r\}$ is such a chain that Ne_r is the direct sum of three simple components.

(b) $\{Ne_1, Ne_2, Ne_3\}$ is such a chain that Ne_2 is the direct sum of three simple components.

(c) $\{Ne_1, Ne_2, Ne_3, Ne_4\}$ is such a chain that Ne_2 is the direct sum of three simple components and Ne_1 and Ne_4 are simple.

(i) First suppose that $\{Ne_1,\cdots,Ne_r\}$ is such a chain as (a). Then an arbitrary indecomposable representation by $\mathfrak{M}=\sum_i\sum_{\lambda_i}Ae_im_{i,\lambda_i}$ has the following form:

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1) James P. Jans [1].
2) James P. Jans [1].
3) In this paper we use the results of [2] without proof.

$$
R_1=\begin{pmatrix}
x_1 \\
& x_2 \\
&& x_3 \\
&&& \ddots \\
&&&& x_r \\
z_{1,1} &&&&& y_1 \\
z_{2,1} & z_{2,2} &&&&& y_2 \\
& z_{3,2} & z_{3,3} &&&&& y_3 \\
&& \ddots & \ddots &&&&& \ddots \\
&&& z_{r-1,r-1} & z_{r-1,r} &&&&& y_{r-1} \\
&&&& z_{r,r} &&&&&& y_r \\
&&&& z_{0,r} &&&&&&& y_0
\end{pmatrix},
$$
[4]

$$
R_2=\begin{pmatrix}
x_1 \\
& x_2 \\
&& x_2 \\
&&& \ddots \\
&&&& x_r \\
&&&&& x_r \\
z_{2,1} & z_{2,2} & 0 &&&& y_2 \\
z'_{2,1} & 0 & z'_{2,2} &&&&& y_2 \\
&& \ddots & \ddots &&&&& \ddots \\
&&&& z_{r-1,r} & 0 &&&& \ddots \\
&&&& 0 & z'_{r-1,r} &&&&& y_{r-1} \\
&&&& z_{r,r} & 0 &&&&&& y_r \\
&&&& 0 & z_{0,r} &&&&&&& y_0
\end{pmatrix},
$$

or

4) The empty place means zero.

$$
R_3=\begin{pmatrix}
x_1 \\
 & x_1 \\
 & & x_2 \\
 & & & x_2 \\
 & & & & \ddots \\
 & & & & & x_r \\
 & & & & & & x_r \\
z_{1,1} & z'_{1,1} & & & & & & y_1 \\
z_{2,1} & 0 & z_{2,2} & & & & & & y_2 \\
0 & z'_{2,1} & 0 & z'_{2,2} & & & & & & y_2 \\
 & & & \ddots & & & & & & & \ddots \\
 & & & & & z_{r-1,r} & 0 & & & & & y_{r-1} \\
 & & & & & 0 & z'_{r-1,r} & & & & & & y_{r-1} \\
 & & & & & z_{r,r} & 0 & & & & & & & y_r \\
 & & & & & 0 & z'_{0,r} & & & & & & & & y_0
\end{pmatrix},
$$

and it is shown by the following way that there exists non singular matrix P such that $PT_1=T_2P$ for arbitrary two indecomposable representations T_1, T_2 which have the same form as R_1, namely

$$
T_1=\begin{pmatrix}
x_1 \\
 & x_2 \\
 & & x_3 \\
 & & & \ddots \\
 & & & & x_r \\
s_{1,1} & & & & & y_1 \\
s_{2,1} & s_{2,2} & & & & & y_2 \\
 & s_{3,2} & & & & & & \ddots \\
 & & \ddots & & & & & & \ddots \\
 & & & & s_{r-1,r} & & & & & y_{r-1} \\
 & & & & s_{r,r} & & & & & & y_r \\
 & & & & s_{0,r} & & & & & & & y_0
\end{pmatrix}
$$

and

$$T_2=\begin{pmatrix}
x_1 &&&&&&&&&& \\
& x_2 &&&&&&&&& \\
&& x_3 &&&&&&&& \\
&&& \ddots &&&&&&& \\
&&&& x_r &&&&&& \\
t_{1,1} &&&&& y_1 &&&&& \\
t_{2,1} & t_{2,2} &&&&& y_2 &&&& \\
& t_{3,2} &&&&&& \ddots &&& \\
&& \ddots &&&&&&&& \\
&&& t_{r-1,r} &&&&& y_{r-1} && \\
&&& t_{r,r} &&&&&& y_r & \\
&&& t_{0,r} &&&&&&& y_0
\end{pmatrix},$$

where $t_{i,j}\neq s_{i,j}$.

Now if we put $P=(p_{i,j})$, we have $p_{i,j}=0$ for $i\lneqq j$ and

$$\begin{aligned}
&t_{1,1}p_{1,1}=p_{r+1,r+1}s_{1,1}, \quad t'_{2,1}p_{1,1}=p_{r+2,r+2}s'_{2,1},\\
&t_{2,2}p_{2,2}=p_{r+2,r+2}s_{2,1}, \quad t_{3,2}p_{2,2}=p_{r+3,r+3}s_{3,2},\\
&\cdots\cdots\cdots\cdots\\
&t_{r,r}p_{r,r}=p_{2r,2r}s_{r,r}, \quad t_{0,r}p_{r,r}=p_{2r+1,2r+1}s_{0,r}.
\end{aligned}$$

Thus there exists P such that $|P|\neq 0$.

By the same way as this computation we can prove it in the case where T_i has the same form as R_1 or R_2.

(ii) Now suppose that $\{Ne_1, Ne_2, Ne_3\}$ is such a chain as (b). Then from the result of [2] an arbitrary indecomposable representation is composed of representations of the following types:

$$R_1=\begin{pmatrix}
x_1 &&&&&&&& \\
& x_1 &&&&&&& \\
&& x_2 &&&&&& \\
&&& x_2 &&&&& \\
z_{1,1} & z'_{1,1} &&& y_1 &&&& \\
z_{2,1} & 0 & z_{2,2} &&& y_2 &&& \\
& z'_{2,1} && z'_{2,2} &&& y_2 && \\
&& z_{3,2} &&&&& y_3 & \\
&&& z'_{4,2} &&&&& y_4
\end{pmatrix}, \qquad
R_2=\begin{pmatrix}
x_1 &&&&&&& \\
& x_2 &&&&&& \\
&& x_2 &&&&& \\
z_{1,1} &&& y_1 &&&& \\
z_{2,1} & z_{2,2} &&& y_2 &&& \\
z'_{2,1} & 0 & z'_{2,2} &&& y_2 && \\
& z_{3,2} &&&&& y_3 & \\
&& z'_{4,2} &&&&& y_4
\end{pmatrix},$$

and
$$R_3=\begin{pmatrix}
x_1 &&&&& \\
& x_2 &&&& \\
z_{1,1} && y_1 &&& \\
z_{2,1} & z_{2,2} && y_2 && \\
& z_{3,2} &&& y_3 & \\
& z_{4,2} &&&& y_4
\end{pmatrix}.$$

For example,

$$T=\begin{pmatrix}
x_1 &&&&&&&&&& \\
& x_1 &&&&&&&&& \\
&& x_2 &&&&&&&& \\
&&& x_2 &&&&&&& \\
&&&& x_3 &&&&&& \\
z_{1,1} &&&&& y_1 &&&&& \\
z_{2,1} & 0 & z_{2,2} & 0 &&& y_2 &&&& \\
0 & z'_{2,1} & 0 & z'_{2,2} &&&& y_2 &&& \\
&& 0 & z'_{3,2} &&&&& y_3 && \\
&& z_{4,2} & 0 & z_{4,3} &&&&& y_4 & \\
&& 0 & z'_{4,2} & z'_{4,3} &&&&&& y_4
\end{pmatrix} \text{ is similar to}$$

$$T'=\begin{pmatrix}
x_3 &&&&&&&&&& \\
& x_1 &&&&&&&&& \\
&& x_2 &&&&&&&& \\
& z_{1,1} && y_1 &&&&&&& \\
& z_{2,1} & z_{2,2} && y_2 &&&&&& \\
z_{4,3} && z_{4,2} &&& y_4 &&&&& \\
&&&&&& x_1 &&&& \\
&&&&&&& x_2 &&& \\
&&&&&& z'_{2,1} & z'_{2,2} & y_2 && \\
&&&&&&& z'_{3,2} && y_3 & \\
z'_{4,3} &&&&&&& z'_{4,2} &&& y_4
\end{pmatrix},$$

where $\begin{pmatrix} x_1 &&&& \\ & x_2 &&& \\ z_{1,1} && y_1 && \\ z_{2,1} & z_{2,2} && y_2 & \\ & z_{4,2} &&& y_4 \end{pmatrix}$ and $\begin{pmatrix} x_1 &&&& \\ & x_2 &&& \\ z'_{2,1} & z'_{2,2} & y_2 && \\ & z'_{3,2} && y_3 & \\ & z'_{4,2} &&& y_4 \end{pmatrix}$ have the same type as R_3.

Then it is shown by the same way as (i) that there exists non singular matrix P such that $PT_1=T_1'P$ where T_1 has the same form as T and T_1' has $u_{i,j}\neq z_{i,j}$ in place of $z_{i,j}$ of T_1.

In the case where T_1 has other form given in [2], this is proved by the same way as above.

(iii) Suppose that $\{Ne_1, Ne_2, Ne_3, Ne_4\}$ is such a chain as (c). Then this is proved by the same way as above.

Thus we have the following

Theorem. *Let $N^2=0$ and k be algebraically closed. Then the class of algebras of bounded representation type and that of algebras of finite representation type are identical.*

References

[1] James P. Jans: On the indecomposable representations of algebras, Dissertation, University of Michigan (1954).

[2] T. Yoshii: On algebras of bounded representation type, Osaka Math. Jour., **8**, No. 1 (1956).