131. On Natural Systems of Some Spaces

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In this note we shall give a brief account about the properties of natural systems and Postnikov's complexes. We state here only the results without proofs.[1] Full details will appear in a forthcoming Journal of the Faculty of Science, Niigata University.

§1. Let X be an arcwise-connected, simply-connected topological space. We shall denote the i-th homotopy group $\pi_i(X, x_0)$ and the natural system of X by π_i and (π_i, k_i) respectively. Let K_i and e^r be the cell-complex of (π_i, k_i) and the unique r-cell of $K_1 = K(\pi_1)$ respectively.

Let $\hat{X}$ be the space of loops on X with x_0 as the end point. Hereafter each notation covered by $\hat{}$ denotes the notation concerned with $\hat{X}$. In particular, $\hat{e}^r$ is the r-dimensional matrix (d_{ij}) where d_{ij} is the unit element of $\hat{\pi}_1$ for each i and j.

In the first place we must note that the following theorem can be proved:

Theorem 1. *$\hat{\pi}_1$ operates trivially on $\hat{\pi}_n$ $(n \geqq 2)$.*

We now define $\rho_{r+1}: \Delta^{r+1} \to \Delta^r \times I$ by

$$\rho_{r+1}(y_1, y_2, \cdots, y_{r+1}) = \begin{cases} (ly_1, ly_2, \cdots, ly_{r+1}), & (y_1+y_2+\cdots+y_r \geqq y_{r+1}), \\ (my_1, my_2, \cdots, my_{r+1}), & (y_1+y_2+\cdots+y_r \leqq y_{r+1}), \end{cases}$$

where $l = \dfrac{y_1+y_2+\cdots+y_{r+1}}{y_1+y_2+\cdots+y_r}$, $m = \dfrac{y_1+y_2+\cdots+y_{r+1}}{y_{r+1}}$ and

$$\Delta^{r+1} = \{(y_1, y_2, \cdots, y_{r+1}): \\ 0 \leqq y_i \leqq 1 \ (i=1,2,\cdots,r+1),\ 0 \leqq y_1+y_2+\cdots+y_{r+1} \leqq 1\}$$

is an $(r+1)$-dimensional Euclidean simplex and Δ^r is the r-face $\Delta^{r+1(r+1)}$ of Δ^{r+1} contained in the hyperplane $y_{r+1}=0$.

Let $\hat{T}^r: \Delta^r \to \hat{X}$ be an r-dimensional singular simplex of $\hat{X}$ and define $\xi_{r+1}: \Delta^r \times I \to X$ by $\xi_{r+1}(P, s) = \hat{T}^r(P)(s)$ where $P \in \Delta^r$ and $s \in I = [0,1]$ is the parameter of loops. Define τ by $\tau\hat{T}^r = \xi_{r+1} \circ \rho_{r+1}: \Delta^{r+1} \to X$. We use the same notation τ for the induced map: $[\hat{T}^r] \to [T^{r+1}]$ subject to the condition that $[\hat{T}^r]$ is an element of $\hat{\pi}_r$, where T^{r+1} is an $(r+1)$-dimensional singular simplex of X. It is easily seen that:

1) τ is an isomorphism of $\hat{\pi}_r$ onto π_{r+1},

1) In this note we quote the notations and definitions from the following report without essential modifications: P. J. Hilton: Report on three papers by M. M. Postnikov (1952).

2) $T^{r+1(i)}=\tau(\hat{T}^{r(i)})$ where (i) denotes the i-th face,

3) $T^{r+1(r+1)}$ is the collapsed map.

Define φ^{r+1}_{i+1*}, $(r+1,\ i+1)$-function over π_{i+1}, by

$$\varphi^{r+1}_{i+1*}(a_0, a_1,\cdots, a_{i+1})=\begin{cases}\tau\psi^r_i(a_0, a_1,\cdots, a_i), & (a_{i+1}=r+1),\\ 0 & , \quad (a_{i+1}<r+1),\end{cases}$$

where ψ^r_i is an (r, i)-function over $\hat{\pi}_i$ and $(a_0, a_1,\cdots, a_{i+1})$ is an $(r+1,\ i+1)$-sequence. And denote by α this transformation from ψ^r_i to φ^{r+1}_{i+1*}. Let $\hat{A}^r_1=(\psi^r_1(i, j))$ be a matrix representation of an r-cell of $\hat{K}_1$, and define α on $\hat{K}_1$ by $\alpha\hat{A}^r_1=(e^{r+1}, \alpha\psi^r_1)$. If α was defined on $\hat{K}_i$, we define α on $\hat{K}_{i+1}$ by $\alpha\hat{A}^r_{i+1}=(\alpha\hat{A}^r_i, \alpha\psi^r_{i+1})$ where $\hat{A}^r_{i+1}=(\hat{A}^r_i, \psi^r_{i+1})$ is an r-cell of $\hat{K}_{i+1}$. Then we see that α is an isomorphism defined on $\hat{K}_i$ for each i. By inductive method we obtain the following:

Theorem 2. *Let X be an arcwise-connected, simply-connected topological space and $\hat{X}$ be the space of loops on X. Then we can construct the natural systems of X and $\hat{X}$ such that the following relations hold for each $i\geqq 3$:*

1) *If $\hat{A}^{r-1}_{i-2}$ is an $(r-1)$-cell of $\hat{K}_{i-2}$, $\alpha\hat{A}^{r-1}_{i-2}$ is an r-cell of K_{i-1}.*

2) $w_{i-1}(\tau\hat{T}^r)=\alpha(\hat{w}_{i-2}\hat{T}^r)$.

3) *The normal $(i-2)$-dimensional singular simplex of $\hat{X}$ corresponding to $(i-2)$-cell $(\cdots((\hat{e}^{i-2}, 0), 0)\cdots, 0)$ is the collapsed map. The normal $(i-1)$-dimensional singular simplex of X corresponding to $(i-1)$-cell $(\cdots((e^{i-1}, 0), 0)\cdots, 0)$ is the collapsed map. If $\hat{T}^{i-2}_N$ is the normal $(i-2)$-dimensional singular simplex of $\hat{X}$ corresponding to $\hat{A}^{i-2}_{i-2}$, $(i-2)$-cell of $\hat{K}_{i-2}$, then $\tau\hat{T}^{i-2}_N$ is the normal $(i-1)$-dimensional singular simplex of X corresponding to $\alpha\hat{A}^{i-2}_{i-2}$.*

4) *The standard $(i-1)$-dimensional singular simplex of $\hat{X}$ corresponding to $(i-1)$-cell $(\cdots((\hat{e}^{i-1}, 0), 0)\cdots, 0)$ is the collapsed map. The standard i-dimensional singular simplex of X corresponding to i-cell $(\cdots((e^{i}, 0), 0)\cdots, 0)$ is the collapsed map. If $\hat{T}^{i-1}_S$ is the standard $(i-1)$-dimensional singular simplex of $\hat{X}$ corresponding to $\hat{A}^{i-1}_{i-2}$, $(i-1)$-cell of $\hat{K}_{i-2}$, then $\tau\hat{T}^{i-1}_S$ is the standard i-dimensional singular simplex of X corresponding to $\alpha\hat{A}^{i-1}_{i-2}$.*

5) $\hat{k}_{i-2}(\cdots((\hat{e}^{i}, 0), 0)\cdots, 0)=0$, $k_{i-1}(\cdots((e^{i+1}, 0), 0)\cdots, 0)=0$, $\hat{k}_{i-2}=\tau^{-1}\circ k_{i-1}\circ\alpha$.

§2. Let X and Y be two arcwise-connected, simply-connected topological spaces, and let $\hat{X}$ and $\hat{Y}$ be the spaces of loops on X and Y respectively. We make the assumption that the natural systems (G_i, k_i), $(\hat{G}_i, \hat{k}_i)$, (H_i, l_i) and $(\hat{H}_i, \hat{l}_i)$ of $X, \hat{X}, Y$ and $\hat{Y}$ have been defined such that they satisfy the relations given in the above Theorem 2. Let K_i, $\hat{K}_i$, L_i and $\hat{L}_i$ be the cell-complexes of the above systems respectively. Assume that (G_i, k_i) and (H_i, l_i) are isomorphic, i.e., there exists, for each i, an isomorphism $\theta_i : G_i \approx H_i$ such that θ_i is a θ_1-isomorphism if $i>1$, and such that there exists for each i a θ_1-isomorphism $\tilde{\theta}_i$ of K_i on L_i, $\tilde{\theta}_i$ being a θ_i-prolongation of $\tilde{\theta}_{i-1}$ with i-cochain d_{i-1}. By inductive method we can prove that $\tilde{\theta}_{i-1}\alpha\hat{K}_i=\alpha\hat{L}_i$. On the assumption mentioned above, we obtain the following:

Theorem 3. $(\hat{G}_i, \hat{k}_i) \approx (\hat{H}_i, \hat{l}_i)$.

Namely, putting $\eta_i=\tau^{-1}\circ\theta_{i+1}\circ\tau$, $\tilde{\eta}_i=\alpha^{-1}\circ\tilde{\theta}_{i+1}\circ\alpha$ and $\hat{d}_{i-1}=\tau^{-1}\circ d_i\circ\alpha$, we can prove that $\eta_i : \hat{G}_i \approx \hat{H}_i$ for each i, η_i is an η_1-isomorphism if $i>1$, $\tilde{\eta}_i$ is an η_1-isomorphism of $\hat{K}_i$ on $\hat{L}_i$ and $\tilde{\eta}_i$ is an η_i-prolongation of $\tilde{\eta}_{i-1}$ with i-cochain $\hat{d}_{i-1}$, for each i. Theorem 3 can be extended in the following form:

Theorem 4. *Let* (G_i, k_i), (G'_i, k'_i), (H_i, l_i) *and* (H'_i, l'_i) *be systems, (not necessarily being natural systems of spaces), and assume that*

1) $G_1=0$, $H_1=0$,

2) G'_1 *operates trivially on* G'_i $(i \geqq 2)$, H'_1 *operates trivially on* H'_i $(i \geqq 2)$,

3) *there exists an isomorphism* τ *such that* $\tau : G'_{i-1} \approx G_i$ *and* $\tau : H'_{i-1} \approx H_i$ $(i \geqq 2)$,

4) $k'_{i-1}=\tau^{-1}\circ k_i \circ \alpha$, $l'_{i-1}=\tau^{-1}\circ l_i \circ \alpha$ *where* α *is the isomorphism defined in* §1, $k_i(\cdots((e^{i+2}, 0), 0)\cdots, 0)=0$ *and* $l_i(\cdots((E^{i+2}, 0), 0)\cdots, 0)=0$ *where* e^{i+2} *and* E^{i+2} *are* $(i+2)$*-dimensional matrices* (d_{mn}) *and* (D_{mn}) *respectively, being* $d_{mn}=1\in G_1$, $D_{mn}=1\in H_1$ *for each* m *and* n,

5) $(G_i, k_i) \approx (H_i, l_i)$.

Then we have $(G'_i, k'_i) \approx (H'_i, l'_i)$.

By making use of Theorem 4 and Postnikov's theorem[2] we obtain the following:

Theorem 5. *Let* (G'_i, k'_i) *be a system such that* $k'_i(\cdots((e'^{i+2}, 0), 0)\cdots, 0)=0$. *Then there exists a space of loops whose natural system is isomorphic to* (G'_i, k'_i) *if and only if* G'_1 *operates trivially on* G'_i $(i \geqq 2)$.

§3. Theorem 4 gives the following fibering theorem:

Theorem 6. *Let* (G_i, k_i) *and* (G'_i, k'_i) *be two systems satisfying the conditions mentioned in Theorem 4. Then there exists a fibering* (E,X,F,p), *in the sense of Serre, such that the natural systems of the base space* X *and of the fiber* F *are isomorphic to* (G_i, k_i) *and* (G'_i, k'_i) *respectively.*

This is a generalization of a fibering of J.-P. Serre.[3]

2) Theorem 3 of the report by P. J. Hilton: Loc. cit.

3) J.-P. Serre: Homologie singulière des espaces fibrés, Ann. Math., **54**, 425–505 (1951).

We have, moreover, a generalization of a fibering of Cartan-Serre:[4)]

Let G_1 and H_1 be multiplicative groups of left operators on abelian groups G_i and H_i ($i \geqq 2$) respectively and assume that the following sequence is exact:

$$\to F_i \underset{f_i}{\to} G_i \underset{g_i}{\to} H_i \underset{h_i}{\to} F_{i-1} \underset{f_{i-1}}{\to} \cdots \underset{g_2}{\to} H_2 \underset{h_2}{\to} F_1 \underset{f_1}{\to} G_1 \underset{g_1}{\to} H_1 \underset{h_1}{\to} 0.$$

Consider two systems (G_i, k_i) and (H_i, l_i), and denote their cell-complexes by K_i and L_i respectively. Assume that the following relations hold:

1) g_i is a homomorphism of G_i onto H_i ($i \geqq 1$),

2) we have $g_i(x_1 x_i) = g_1(x_1) g_i(x_i)$, for all elements $x_1 \in G_1$ and $x_i \in G_i$,

3) $g_{i+1} \circ k_i = l_i \circ \bar{g}_i$, defining $\bar{g}_1 : K_1 \to L_1$ by $\bar{g}_1(d_{ij}) = (g_1(d_{ij}))$ and $\bar{g}_i$ on K_i by $\bar{g}_i A_i^r = (\bar{g}_{i-1} A_{i-1}^r, g_i \circ \varphi_i^r)$ for each r-cell $A_i^r = (A_{i-1}^r, \varphi_i^r)$ of K_i inductively.

Then we see that $\bar{g}_i(K_i) \subset L_i$ ($i \geqq 2$).

After these preparations we can obtain the following theorem, by making use of Postnikov's theorem,[2)] in the same way as the fibering of Cartan-Serre:[4)]

Theorem 7. *On the assumptions mentioned above, there is a fibering (E, X, F, p) in the sense of Serre, such that the natural systems of the fiber space E and of the base space X are isomorphic to (G_i, k_i) and (H_i, l_i) respectively and the homotopy exact sequence of E is isomorphic to the exact sequence given above.*

4) J.-P. Serre: Cohomologie modulo 2 des complexes d'Eilenberg-MacLane, Comment. Math. Helv., **27**, 198–232 (1953).