123. *Fourier Series. II. Order of Partial Sums*

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1. **Introduction.** Let $f(t)$ be an integrable function with period 2π and its Fourier series be

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nt+b_n \sin nt).$$

By $s_n(x)$ we denote the nth partial sum of the Fourier series. We put as usual $\varphi_x(u)=f(x+u)+f(x-u)$.

H. Lebesgue [1] has proved the following

Theorem 1. *If, for a fixed x,*

$$\int_0^t |\varphi_x(u)|\,du=o(t) \tag{1}$$

as $t\to 0$, then

$$s_n(x)=o(\log n). \tag{2}$$

S. Izumi [2] proved

Theorem 2. *The conditions*

$$\int_0^t \varphi_x(u)\,du=o(t), \tag{3}$$ $$\int_0^t |\varphi_x(u)|\,du=O(t) \qquad (t\to 0) \tag{4}$$

do not imply (2) in general.

Then there arises the problem: What condition with (3) does (2) imply? As an answer of this problem we prove the following

Theorem 3. *If (3) holds and*

$$\int_0^t (f(\xi+u)-f(\xi-u))\,du=o(t) \qquad (t\to 0) \tag{5}$$

uniformly in ξ in a neighbourhood of x, then (2) holds.

This is proved by the same idea as in the proof of Theorem 7 in [4].

On the other hand O. Szász [3] proved that:

Theorem 4. *If*

$$\int_0^t |\varphi_x(u)|\,du=o\left(t\Big/\log\frac{1}{t}\right) \qquad (t\to 0), \tag{6}$$

then

$$s_n(x)=o(\log\log n). \tag{7}$$

Analogously as Theorems 2 and 3, we prove the following theorems.

Theorem 5. *The conditions*

$$\int_0^t \varphi_x(u)\,du=o\left(t\Big/\log\frac{1}{t}\right), \tag{8}$$

$$(9) \qquad \int_0^t |\varphi_x(u)|\,du = O\left(t \Big/ \log\frac{1}{t}\right)$$

do not imply (7) in general.

Theorem 6. *If (8) holds and*

$$(10) \qquad \int_0^t (f(\xi+u)-f(\xi-u))\,du = o\left(t \log\log\frac{1}{t} \Big/ \log\frac{1}{t}\right) \qquad (t\to 0)$$

uniformly in a neighbourhood of x, then (7) holds.

This may be proved similarly as Theorem 3, and we omit its proof.

2. Proof of Theorem 3. We have

$$s_n(x) = \frac{1}{\pi}\int_0^\pi \varphi_x(t)\frac{\sin nt}{t}\,dt + o(1)$$
$$= \frac{1}{\pi}\left[\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right] + o(1) = \frac{1}{\pi}[I+J] + o(1).$$

By integration by parts, we get

$$|I| \leqq \int_0^{\pi/n} |\varPhi_x(t)| \left|\frac{\sin nt}{t^2} - \frac{n\cos nt}{t}\right| dt = o\left(n\int_0^{\pi/n} dt\right) = o(1),$$

where $\varPhi_x(t) = \int_0^t \varphi_x(u)\,du = o(t)$ as $t\to 0$. We now write

$$J = \int_{\pi/n}^{\pi} \varphi_x(t)\frac{\sin nt}{t}\,dt = J_1 - J_2 + o(1),$$

where

$$J_1 = \sum_{k=1}^{(n-1)/2}\int_0^{\pi/n} \frac{\varphi_x(t+2k\pi/n)-\varphi_x(t+(2k-1)\pi/n)}{t+2k\pi/n}\sin nt\,dt,$$

$$J_2 = \sum_{k=1}^{(n-1)/2}\int_0^{\pi/n} \varphi_x(t+(2k-1)\pi/n)\left(\frac{1}{t+2k\pi/n} - \frac{1}{t+(2k-1)\pi/n}\right)\sin nt\,dt$$

and further we divide J_1 into two parts as follows:

$$J_1 = \int_0^{\pi/n} \frac{\varphi_x(t+2k\pi/n)-\varphi_x(t+(2k-1)\pi/n)}{2k\pi/n}\sin nt\,dt$$
$$-\int_0^{\pi/n} \frac{\varphi_x(t+2k\pi/n)-\varphi_x(t+(2k-1)\pi/n)}{(t+2k\pi/n)2k\pi/n}\,t\sin nt\,dt$$
$$= J_{11} - J_{12}.$$

We shall first estimate J_{11}. We have

$$J_{11} = \sum_{k=1}^{(n-1)/2}\frac{n}{2k\pi}\left[\int_0^{\pi/n}(f(x+t+2k\pi/n)-f(x+t+(2k-1)\pi/n))\sin nt\,dt\right.$$
$$\left.-\int_0^{\pi/n}(f(x-t-2k\pi/n)-f(x-t-(2k-1)\pi/n))\sin nt\,dt\right]$$
$$= \sum_{k=1}^{(n-1)/2}\frac{n}{2k\pi}[J_{11}^1 + J_{11}^2]$$

and

$$J_{11}^1=\int_0^{\pi/2n}(f(x+2k\pi/n)-f(x+(2k-1)\pi/n-t))\sin nt\,dt$$

$$+\int_0^{\pi/2n}(f(x+2k\pi/n+(\pi/n-t))-f(x+(2k-1)\pi/n+(\pi/n-t)))\sin nt\,dt$$

$$=\int_0^{\pi/2n}(f(\xi+t)-f(\xi-t))\sin nt\,dt-\int_{\pi/2n}^{\pi/n}(f(\xi+t)-f(\xi-t))\sin nt\,dt,$$

where $\xi=x+2k\pi/n$, $\tau=t-\pi/n$. By integration by parts and by the condition (5)

$$\int_0^{\pi/2n}(f(\xi+t)-f(\xi-t))\sin nt\,dt=\Big[\sin nt\int_0^t(f(\xi+u)-f(\xi-u))\,du\Big]_0^{\pi/2n}$$

$$-n\int_0^{\pi/2n}\cos nt\,dt\int_0^t(f(\xi+u)-f(\xi-u))\,du$$

$$=o(1/n)+o\Big(n\int_0^{\pi/2n}t\,dt\Big)=o(1/n),$$

and also

$$\int_0^{\pi/n}(f(\xi+t)-f(\xi-t))\sin nt\,dt=o(1/n).$$

Hence $\sum_{k=1}^{(n-1)/2}\frac{n}{2k\pi}J_{11}^1=o\,(\log n)$.

Similarly we also get $\sum_{k=1}^{(n-1)/2}\frac{n}{2k\pi}J_{11}^2=o\,(\log n)$, and thus we have proved that $J_{11}=o\,(\log n)$.

On the other hand we write

$$J_{12}=\sum_{k=1}^{(n-1)/2}\frac{n}{2k\pi}\Big[\int_0^{\pi/n}\frac{f(x+t+2k\pi/n)-f(x+t+(2k-1)\pi/n)}{t+2k\pi/n}t\sin nt\,dt$$

$$+\int_0^{\pi/n}\frac{f(x-t-2k\pi/n)-f(x-t-(2k-1)\pi/n)}{t+2k\pi/n}t\sin nt\,dt\Big]$$

$$=\sum_{k=1}^{(n-1)/2}\frac{n}{2k\pi}[J_{12}^1+J_{12}^2].$$

Hence by integration by parts and by the condition (5), we have

$$J_{12}^1=-\int_0^{\pi/n}F_x(t)\frac{(2k\pi/n)\sin nt+nt^2\cos nt+2k\pi t\cos nt}{(t+2k\pi/n)^2}dt$$

and hence

$$\sum_{k=1}^{(n-1)/2}\frac{n}{2k\pi}J_{12}^1=\sum_{k=1}^{(n-1)/2}\Big(\frac{n}{k}\Big)^3\int_0^{\pi/n}o\Big(\frac{1}{n}\Big)(nt^2+4tk)\,dt=o(1),$$

where $F_x(t)=\int_0^t(f(x+u+2k\pi/n)-f(x+u+(2k-1)\pi/n))du=o(1/n)$ uniformly for x and k as $n\to\infty$ $(0\leqq t\leqq\pi/n)$. In the same way we get $\sum_{k=1}^{(n-1)/2}\frac{n}{2k\pi}J_{12}^2=o(1)$, and we thus have $J_{12}=o(1)$.

Finally we shall prove $J_2=o\,(\log n)$. By Abel's lemma

$$J_2=\sum_{k=1}^{(n-1)/2}\int_0^{\pi/n}\sum_{j=k}^{n}\left(\frac{1}{t+2j\pi/n}-\frac{1}{t+(2j-1)\pi/n}\right)$$
$$\cdot(\varphi_x(t+(2k-1)\pi/n)-\varphi_x(t+(2k-3)\pi/n))\sin nt\,dt$$
$$+\int_0^{\pi/n}\sum_{j=1}^{n}\left(\frac{1}{t+2j\pi/n}-\frac{1}{t+(2j-1)\pi/n}\right)\varphi_x(t+\pi/n)\sin nt\,dt$$
$$=J_{21}+J_{22},$$

say. Then by integration by parts

$$J_{21}=-\sum_{k=1}^{(n-1)/2}\frac{\pi}{n}\int_0^{\pi/n}\sum_{j=k}^{n}\left(\frac{n\cos nt}{(t+2j\pi/n)(t+(2j-1)\pi/n)}\right.$$
$$\left.-\frac{\sin nt\,(2t+(4j-1)\pi/n)}{(t+2j\pi/n)^2(t+(2j-1)\pi/n)^2}\right)$$
$$\cdot\int_0^t(\varphi_x(u+(2k-1)\pi/n)-\varphi_x(u+(2k-3)\pi/n))\,du,$$

whence

$$J_{21}=\sum_{k=1}^{(n-1)/2}\frac{\pi}{n}\int_0^{\pi/n}\sum_{j=k}^{n}\left(\frac{n^3}{j^2}+\frac{n^3}{j^3}\right)o\left(\frac{1}{n}\right)dt=o\,(\log n)$$

by the condition (5). Furthermore, we have also by integration by parts

$$J_{22}=-\frac{\pi}{n}\int_0^{\pi/n}\sum_{j=1}^{n}\left(\frac{n\cos nt}{(t+2j\pi/n)(t+(2j-1)\pi/n)}\right.$$
$$\left.-\frac{(2t+(4j-1)\pi/n)\sin nt}{(t+2j\pi/n)^2(t+(2j-1)\pi/n)^2}\right)dt\int_0^t\varphi_x(u+\pi/n)\,du$$

and then applying the condition (3)

$$|J_{22}|\leqq A\frac{1}{n}\sum_{j=1}^{n}\frac{n^3}{j^2}\int_0^{\pi/n}dt\left|\int_0^t(f(x+u+\pi/n)+f(x-u-\pi/n))\,du\right|$$
$$\leqq An^2\int_0^{\pi/n}dt\left|\left[\int_0^{t+\pi/n}-\int_0^{\pi/n}-\int_{-\pi/n}^{0}+\int_{-\pi/n-t}^{0}\right]f(x+u)\,du\right|=o(1)$$

where A is an absolute constant. Thus the theorem is proved.

3. Proof of Theorem 5. Let

$$\mu_1=3,\quad \mu_k=3^{2\mu_{k-1}}\quad(k=2,3,\cdots)$$

and $f(t)$ is an even function such that

$$f(t)=\varepsilon_j^k\Big/\log\frac{\mu_k}{2j\pi}\qquad\text{in }((2j-1)\pi/\mu_k<t<2j\pi/\mu_k),$$
$$=-\varepsilon_j^k\Big/\log\frac{\mu_k}{2j\pi}\quad\text{in }(2j\pi/\mu_k<t<(2j+1)\pi/\mu_k),$$

where $j=1,2,\cdots,((\mu_k/\mu_{k-1})-1)/2$, $k=1,2,\cdots$ and $0<\varepsilon_j^k\leqq 1$, $\varepsilon_j^k\to 0$ as $k\to\infty$ for each j. For example, we take

$$\varepsilon_j^k=1/k\ (1\leqq j\leqq k),\quad \varepsilon_j^k=1\ (j>k).$$

Then, if $\pi/\mu_k<t<\pi/\mu_{k-1}$, then there is a j such that

$$(2j-1)\pi/\mu_k<t<(2j+1)\pi/\mu_k.$$

For such t

$$\int_0^t f(u)du = \int_{\pi/\mu_k}^t f(u)du + \sum_{l=k}^{\infty} \int_{\pi/\mu_{l+1}}^{\pi/\mu_l} f(u)du$$

where

$$\int_{\pi/\mu_{l+1}}^{\pi/\mu_l} f(u)du = 0$$

and

$$\left| \int_{\pi/\mu_k}^t f(u)du \right| = \left| \int_{(2j-1)\pi/\mu_k}^t f(u)du \right|$$

$$\leqq \frac{\pi}{\mu_k} \frac{\varepsilon_j^k}{\log(\mu_k/2j\pi)} = \frac{\varepsilon_j^k}{2j} \left(\frac{2j\pi/\mu_k}{\mu_k \log(\mu_k/2j\pi)} \right).$$

Thus we have

$$\int_0^t f(u)du = o\left(t \Big/ \log\frac{1}{t}\right) \qquad (t \to 0).$$

We have also

$$\int_0^t |f(u)|\,du = A \int_0^t \frac{du}{\log(2\pi/u)} \leqq At \Big/ \log\frac{2\pi}{t}.$$

On the other hand

$$s_n(0) = \frac{2}{\pi} \int_0^{\pi} f(t) \frac{\sin nu}{u} du + o(1).$$

For $n = \mu_k$, we put

$$s_{\mu_k}(0) = \frac{2}{\pi} \left(\int_0^{\pi/\mu_k} + \int_{\pi/\mu_k}^{\pi/\mu_{k-1}} + \int_{\pi/\mu_{k-1}}^{\pi} \right) + o(1)$$

$$= \frac{2}{\pi} [I + J + K] + o(1).$$

We have then

$$|I| \leqq \int_0^{\pi/\mu_k} \frac{\mu_k}{\log 1/u} du = o(1),$$

$$J = \int_{\pi/\mu_k}^{\pi/\mu_{k-1}} f(u) \frac{\sin \mu_k u}{u} du = -\int_{\pi/\mu_k}^{\pi/\mu_{k-1}} |f(u)| \frac{|\sin \mu_k u|}{u} du,$$

and then

$$|J| \geqq A \int_{\pi k/\mu_k}^{\pi/\mu_{k-1}} \frac{du}{u \log(2\pi/u)} \geqq A \log\log \mu_k.$$

Finally

$$|K| \leqq \int_{\pi/\mu_{k-1}}^{\pi} \frac{|f(u)|}{u} du \leqq A \log \mu_{k-1} \leqq A \log\log\log \mu_k.$$

Accordingly

$$|s_{\mu_k}(0)| \geqq A \log\log \mu_k,$$

i.e. $s_n(0) \neq o(\log\log n)$ for infinitely many n. Thus the theorem is proved.

References

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