150. Fourier Series. IV. Korevaar's Conjecture

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1. J. Korevaar [1] has proved the following theorem.

Theorem 1. Let $f(x)$ be a periodic function with period 2π which is continuous except for a finite number of jump discontinuities and which belongs to the class Lip 1 on every subinterval where $f(x)$ is continuous. Then there is a constant A_1, depending on the Lipschitz constants, supremum of the absolute value of $f(x)$ and the set of jump points, such that for every n there is a trigonometrical polynomial

$$s_n(x)=\sum_{k=0}^{n}(a_k\cos kx+b_k\sin kx) \tag{1}$$

of order n, which satisfies

$$\int_{-\pi}^{\pi}|f(x)-s_n(x)|\,dx < A_1/n, \tag{2}$$

$$|a_k|<A_1/k,\quad |b_k|<A_1/k\quad (k=0,1,\cdots,n). \tag{3}$$

We can see, as an immediate consequence of a result due to A. Zygmund [2], that the left side of (2) can not be $o(1/n)$ as $n\to\infty$. In this connection J. Korevaar surmised the truth of the following

Conjecture. Let $f(x)$ be a periodic function with period 2π which is continuous except jump discontinuity at a point ξ and belongs to the class Lip 1 in the interval $(\xi,\xi+2\pi)$. Then there is a constant A_2 such that for every n and every trigonometrical polynomial $t_n(x)$ of order n

$$\int_{-\pi}^{\pi}|f(x)-t_n(x)|\,dx > A_2/n. \tag{4}$$

We shall here prove this.

2. Proof of the conjecture. Let

$$E_n(f;-\pi,\pi)=\min_{(t_n)}\int_{-\pi}^{\pi}|f(x)-t_n(x)|\,dx,$$

where the minimum is taken for all trigonometrical polynomial $t_n(x)$ of order n. It is sufficient to prove that

$$E_n(f;-\pi,\pi)\geqq A_2/n.$$

Evidently

$$E_n(f;-\pi,\pi)\geqq E_n(f;-\varepsilon\pi/n,\varepsilon\pi/n),$$

where ε is a positive number which will be determined later. If we put $t_n(x)=\sum_{k=0}^{n}(a_k\cos kx+b_k\sin kx)$, then

$$\int_{-\varepsilon\pi/n}^{\varepsilon\pi/n} |f(x)-t_n(x)|\,dx \geqq \int_{-\varepsilon\pi/n}^{\varepsilon\pi/n} \left| f(x)-\sum_{k=1}^{n} a_k - x\sum_{k=1}^{n} k\, b_k \right| dx$$

$$-\int_{-\varepsilon\pi/n}^{\varepsilon\pi/n} \left| \sum_{k=1}^{n} a_k(1-\cos kx) \right| dx - \int_{-\varepsilon\pi/n}^{\varepsilon\pi/n} \left| \sum_{k=1}^{n} b_k(kx-\sin kx) \right| dx$$

$$=I-J-K.$$

We can easily see that

$$I \geqq A_3\, \varepsilon\pi/n,$$

$$J \leqq \frac{1}{2}\int_{-\varepsilon\pi/n}^{\varepsilon\pi/n} \sum_{k=1}^{n} |a_k|\, k^2 x^2\, dx$$

$$= \frac{1}{2}\left(\frac{\varepsilon\pi}{n}\right)^3 \sum_{k=1}^{n} |a_k|\, k^2,$$

$$K \leqq \frac{1}{3}\int_{-\varepsilon\pi/n}^{\varepsilon\pi/n} \sum_{k=1}^{n} |b_k|\, k^3 x^3\, dx$$

$$= \frac{1}{6}\left(\frac{\varepsilon\pi}{n}\right)^4 \sum_{k=1}^{n} b_k\, k^3.$$

Now, let $s_n(x)$ be a polynomial satisfying the condition in Theorem 1, then

$$\int_{-\pi}^{\pi} |f(x)-s_n(x)|\,dx < A_1/n,$$

$$(5)\quad \int_{-\pi}^{\pi} |f(x)-t_n(x)|\,dx \geqq \int_{-\pi}^{\pi} |s_n(x)-t_n(x)|\,dx - \int_{-\pi}^{\pi} |f(x)-s_n(x)|\,dx.$$

We write

$$s_n(x)-t_n(x) = \sum_{k=0}^{n} (\alpha_k \cos kx + \beta_k \sin kx)$$

and we shall apply Theorem 1. Then

$$\max(|\alpha_k|,\ |\beta_k|) \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} |s_n(x)-t_n(x)|\,dx.$$

If we take the $t_n(x)$ which minimizes the left side of (5), then

$$\max(|\alpha_k|,\ |\beta_k|) \leqq \frac{1}{\pi}\left(E_n(f;\ -\pi,\pi) + \int_{-\pi}^{\pi} |f(x)-s_n(x)|\,dx\right)$$

$$\leqq \frac{1}{\pi}\left(\frac{A_1}{n}+\frac{A_1}{n}\right) = 2A_1/\pi n,$$

and hence

$$\max(|a_k|,\ |b_k|) \leqq A_4/k \quad (k=1,2,\cdots,n).$$

Thus we have

$$J \leqq \frac{1}{3}\left(\frac{\varepsilon\pi}{n}\right)^3 \frac{A_4}{n} \sum_{k=1}^{n} k^2 = \frac{\varepsilon^3 A_5}{n},$$

$$K \leqq \frac{1}{6}\left(\frac{\varepsilon\pi}{n}\right)^4 \frac{A_4}{n} \sum_{k=1}^{n} k^3 = \frac{\varepsilon^4 A_6}{n}.$$

Therefore

$$E_n(f;\ -\pi,\ \pi) \geqq \frac{\varepsilon}{n}(A_3\pi - A_5\varepsilon^2 - A_6\varepsilon^3).$$

If we take ε such that

$$A_3\pi/2 > A_5\varepsilon^2 + A_6\varepsilon^3,$$

then we get

$$E_n(f;\ -\pi,\ \pi) \geqq \frac{A_3\pi\varepsilon}{2}\frac{1}{n} = A_2/n,$$

which is the required.

3. Finally we remark that the conjecture holds when $f(t)$ has a finite number of jump discontinuities. This may easily be seen from §2.

References

[1] J. Korevaar: Proc. Amsterdam Acad., **56** (1951).
[2] A. Zygmund: Duke Math. Journ., **10** (1943).