7. Contributions to the Theory of Semi-groups. VI

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(Comm. by K. KUNUGI, M.J.A., Jan. 12, 1957)

In this Note, we shall give some supplement remarks of my papers [1, I–V]. A proposition proved is a generalisation of a theorem by S. Schwarz [2].

By a *character* of semi-group S *) we mean a complex valued function $\chi(x)$ satisfying $\chi(a)\chi(b)=\chi(ab)$ for every a, b of S.

The set $\hat{S}$ of all characters of S is a *commutative semi-group with zero and unit*. For χ, ψ of $\hat{S}$, the product $\chi\psi$ is defined as $\chi\psi(a)=\chi(a)\psi(a)$ for all a of S.

Let $\mathfrak{A}$ be an ideal of S, then the set $\hat{\mathfrak{A}}$ of all elements χ of $\hat{S}$ such that $\chi(x)=0$ for $x \in \mathfrak{A}$ is an ideal of $\hat{S}$. Clearly $\hat{\mathfrak{A}}$ is not *empty* and *closed*.

Conversely, if S is a periodic semi-group with finite numbers of idempotents, for every proper ideal $\hat{\mathfrak{A}}$ of $\hat{S}$, the set $\hat{\hat{\mathfrak{A}}}$ of all elements x consisting of $\chi(x)=0$ for all $\chi \in \hat{\mathfrak{A}}$ is non-empty and an ideal of S.

Let $\mathfrak{A}$ be a closed ideal in S, then the ideal $\mathfrak{A}$ is the intersection of some prime ideals $\mathfrak{P}_\lambda$ i.e. $\mathfrak{A}=\bigcap_\lambda \mathfrak{P}_\lambda$. Therefore,

$$\varepsilon_\lambda(x)=\begin{cases} 0 & x \in \mathfrak{P}_\lambda \\ 1 & x \in S-\mathfrak{P}_\lambda \end{cases}$$

are in $\hat{S}$ and each $\varepsilon_\lambda(x)$ is contained in $\hat{\mathfrak{A}}$. Then we have $\hat{\hat{\mathfrak{A}}}=\mathfrak{A}$. Therefore in such a semi-group S, there is a *one-to-one correspondence between the closed ideals in S and the ideals of $\hat{S}$.*

Let $\mathfrak{A}, \mathfrak{B}$ be two closed ideals and let $\mathfrak{A}\subset\mathfrak{B}$, then we have $\hat{\mathfrak{A}}\supseteq\hat{\mathfrak{B}}$. To prove $\hat{\mathfrak{A}}\supset\hat{\mathfrak{B}}$, by the Zorn lemma, we take a maximal subsemi-group M such that $M\subset\mathfrak{B}$ and $\mathfrak{A}\frown M=\phi$. By using the Zorn lemma again, we find a maximal ideal $\mathfrak{M}$ such that $\mathfrak{A}\subset\mathfrak{M}$ and $\mathfrak{A}\frown\mathfrak{M}=\phi$. Then since $\mathfrak{M}$ is a prime ideal, we can define a character χ such that

$$\chi(x)=\begin{cases} 1 & x \in \mathfrak{M} \\ 0 & x \in S-\mathfrak{M}. \end{cases}$$

Then $\chi \in \hat{\mathfrak{A}}$, and, from $\chi(x)=1$ for $x \in \mathfrak{B}$, $\chi \in \hat{\mathfrak{B}}$.

Thus we have the following

Proposition. In any commutative periodic semi-group having a finite number of idempotents, there is a one-to-one correspondence

*) For undefined terminologies, see my Notes [I–V].

between the closed ideals in S and the ideals in $\hat{S}$, *and* $\mathfrak{A} \subset \mathfrak{B}$ *in S if and only if* $\hat{\mathfrak{A}} \supset \hat{\mathfrak{B}}$.

References

[1] K. Iséki: Contributions to the theory of semi-groups. I–V, Proc. Japan Acad., **32**, 174–175, 225–227, 323–324, 430–435, 560–561 (1956).
[2] S. Schwarz: O nekotoroi sbyazi Galois b Teorii Karakterob polugrupp, Czechoslovak Math. Jour., **4** (79), 296–313 (1954).