16. Certain Subgroup of the Idèle Group

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Let k be an algebraic number field of finite rank over the rational number field Q, I the group of idèles of k, P the group of principal idèles of k, C the idèle class group I/P, H' the maximal compact subgroup in the connected component H of the unit element of I, D' the natural image (isomorphic) of H' into C, and D the connected component of the unit element of C. Clearly $D\supset D'$, and D/D' is, as shown by Weil in his article [5], an infinitely and uniquely divisible group.[1] Combining it with Grunwald's lemma corrected by Wang and Hasse,[2] we shall prove in the present article the following

Theorem. Let J be the subgroup in I consisting of all of such idèles each of which has 1 as its component at every prime divisor of k except a nulset (with reference to Kronecker density) of finite prime divisors of k. Then, the natural homomorphism ν of J into C/D is an isomorphism.

We prepare two lemmas. Let n be a natural number, ς_{2^n} a primitive 2^n-th root of 1, $L_n=Q(\varsigma_{2^n})\frown k$. Clearly, there exists a natural number N' such that for every n greater than N', $L_n=L_{N'}$. Let $N=N'+3$. Then, it holds the following

Lemma 1. Let l be a natural prime number and n a natural number greater than M_l, where $M_l=1$ for $l\neq 2$ and $M_l=N$ for $l=2$. Let α be a number in k such that α is l^n-th power residue at every prime divisor of k except a nulset (with reference to Kronecker density) of prime divisors of k. Then, α is l^{n-1}-th power of a number in k.

Proof. When $\alpha=0$, the lemma is trivial. Let α be a non zero number in k satisfying the condition of the lemma. Then, there exists a set T of finite prime divisors of k with 1 as its Kronecker density such that for each $p\in T$, α is l^n-th power of an element in the completion field k_p of k for p. So, α is l^n-th power of a number in $k(\varsigma)$, where ς is a primitive l^n-th root of 1. Then, α is, from Theorem 1 (Satz 1) in Hasse's article [3], l^n-th power of a number in k, if $l\neq 2$, and α is from the supposition for N and from Theorem 2 (Satz 2) in the above quoted article [3], l^{n-1}-th power of a number in k, even if $l=2$, and we obtain the lemma.

Lemma 2. Let p be a finite prime divisor of k, α a non zero

1) Cf. [1].
2) Cf. [2], [3], [4], esp. [3].

element in the completion field k_p of k for p such that for every natural number n and for every prime natural number l, a is always l^n-th power of an element in k_p. Then, $a=1$.

Proof. Let a be a non zero element in k_p satisfying the condition in the lemma. Then a is clearly a unit. As is well known, the multiplicative group U_p of the units of k_p is isomorphic with a Galois group ($G(A_p/Z_p)$ in the following), and $a=1$, q.e.d.

Proof of Theorem. Let a be an idèle in J such that $\nu(a) \in D$. As D/D' is infinitely divisible, there exist for each natural number n and for each natural prime number l an idèle $b_{l,n}$ and a non zero number $\alpha_{l,n}$ in k such that

$$ab_{l,n}l^n\alpha_{l,n} \in H'. \qquad (1)$$

So, $\alpha_{l,n}$ is l^n-th power residue for every finite prime divisor of k except a nulset of prime divisors of k. Suppose that n is sufficiently large. Then, α is from Lemma 1 l^{n-1}-th power of a non zero element in k. So, each of the local components $\iota_p(a)$ of a for every finite prime divisor p of k is l^{n-1}-th power of an element in k_p. As l is arbitrary prime natural number and n is arbitrarily large natural number, it follows from Lemma 2, that $\iota_p(a)=1$ and $a=1$, q.e.d.

Corollary 1. Let A_p be a maximal abelien extension of a completion field of k for a prime divisor p of k, μ an injection of a maximal abelien extension A of k into A_p/k_p. Then, $k_p\mu(A)=A_p$.

Proof. Let Z_p be the maximum subfield in A_p without ramification over k_p. As is well known, $Z_p \subset k_p\mu(A)$. Let φ_p be the local norm residue symbol of k_p. Obviously, φ_p is an isomorphism of the multiplicative group k_p^* of the non zero elements in k_p into the Galois group $G(A_p/k_p)$, and it maps the subgroup U_p of the units in k_p onto the Galois group $G(A_p/Z_p)$ of A_p over Z_p. Let φ be the global norm residue symbol of k. Obviously, φ is homomorphism of I onto the Galois group $G(A/k)$ of A over k having $\overline{D}$ as its kernel, where $\overline{D}$ is the subgroup in I consisting of idèles involved in elements in D. It follows from the above theorem that φ, restricted into k_p^* (involved in I), gives an isomorphism of k_p^* into the Galois group $G(A/k)$. So $\varphi\varphi_{p-1}$ gives an isomorphism of $G(A_p/Z_p)$ into $G(A/k)$ and the restriction of $G(A_p/Z_p)$ into $k_p\mu(A)$ gives an isomorphism of $G(A_p/Z_p)$ onto $G(k_p\mu(A)/Z_p)$, which certifies the corollary.

As $\varphi_p(k_p^*)$ is dense in $G(A_p/k_p)$, we obtain easily the following

Corollary 2. Let K_p be a finite extension in A_p. Then, there exists a finite extension K of k in A having an injection μ such that $K_p=k_p\mu(K)$.

Let B be a Galois extension field of k, involving A and having an injection μ into A_p/k_p. As $G(A_p/k_p)$ is a completion of $\varphi_p(k_p^*)$ and k_p^* is locally compact abelien group, it follows from the above theorem and Corollary 1, that μ induces canonically an isomorphism μ^* of $G(A_p/k_p)$ into $G(B/k)$, and we obtain easily the following corollaries.

Corollary 3. The restriction of $\mu^*(G(A_p/k_p))$ into A/k induces an isomorphism of $\mu^*(G(A_p/k_p))$ into $G(A/k)$.

Corollary 4. Let B be a Galois extension of k involving A such that every valuation of B is obtained by an injection of B into A_p/k_p, i.e. B is everywhere locally abelien. Then, the intersection of the commutator group of $G(B/k)$ with the union of the Galois groups $\mu^*(A_p/k_p)$ of the decomposition fields of non archimedien valuations of B consists only of the identity.

Remark. It is known by an example (construction of Scholz) that there exists a finite algebraic number field k having $B \supsetneqq A_k$ satisfying the condition of Corollary 4.

References

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