15. Fourier Series. XII. Bernstein Polynomials

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1. If $f(t)$ is integrable in the closed interval $[0,1]$, then the generalized Bernstein polynomials of $f(t)$ are defined as

$$(1)\qquad P_n(x)=P_n(x,f)=\sum_{\nu=0}^{n}(n+1)p_{n,\nu}(x)\int_{\nu/(n+1)}^{(\nu+1)/(n+1)}f(t)dt \qquad (n=0,1,2,\cdots),$$

where

$$(2)\qquad p_{n,\nu}(x)=p_{n,\nu}=\binom{n}{\nu}x^\nu(1-x)^{n-\nu}.$$

It is known that $P_n(x,f)$ tends to $f(x)$ almost everywhere as $n\to\infty$ and carries many properties of the Fejér mean of the Fourier series of $f(t)$ [1]. From this point of view P. L. Butzer [2] considered the polynomials, corresponding to the partial sums of the Fourier series of $f(t)$ such that

$$(3)\qquad Q_n(x)=Q_n(x,f)=(n+1)P_n(x,f)-nP_{n-1}(x,f) \qquad (n=0,1,2,\cdots),$$

and established some fundamental theorems concerning them.

Among others he proved the following

Theorem 1. *If $f(t)$ is bounded in the interval $(0,1)$ and its second derivative exists at $t=x$, then $Q_n(x,f)$ tends to $f(x)$ as $n\to\infty$.*

Further he raised the question:

Does there exist an integrable function $f(t)$ such that the $Q_n(x,f)$ diverges almost everywhere in the interval $(0,1)$?

In the present note we wish to prove the following theorems:

Theorem 2. *If the derived Fourier series of $f(t)$ converges absolutely, then $Q_n(x,f)$ converges to $f(x)$ everywhere.*

Theorem 3. *There is a continuous function $f(t)$ with absolutely convergent Fourier series such that $Q_n(x,f)$ diverges almost everywhere.*

Clearly Theorem 3 is a stronger solution of the problem of Butzer's. We note that, as will be found incidentally in § 3, our Theorem 2 can not hold in general unless the derived Fourier series of $f(t)$ is absolutely convergent.

2. **Proof of Theorem 2.** Without loss of generality we may suppose that

$$f(t)\sim\sum_{\lambda=1}^{\infty}a_\lambda e^{2\pi i\lambda t}$$

Then

$$(4)\qquad Q_n(x,f)-f(x)=\sum a_\lambda[Q_n(x,e^{2\pi i\lambda t})-e^{2\pi i\lambda x}]$$

Since we can easily see from [1, p. 21] that*)

$$|Q_n(x, e^{2\pi i\lambda t})-e^{2\pi i\lambda x}|\leqq A\lambda,$$

(4) converges absolutely by the assumption that $\sum \lambda|a_\lambda|<\infty$.

On the other hand

$$Q_n(x, e^{2\pi i\lambda t})-e^{2\pi i\lambda x}\to 0 \qquad (n\to\infty) \tag{5}$$

for all fixed λ.

Let ε be any positive number. Then there is an N such that $\sum_{\lambda=N+1}^{\infty}\lambda|a_\lambda|<\varepsilon$, and hence by (5)

$$\limsup_{n\to\infty}|Q_n(x, f)-f(x)|\leqq\varepsilon.$$

Thus we get Theorem 2.

3. Proof of Theorem 3. Let us set

$$f(t)\sim\sum_{\lambda=1}^{\infty}a_\lambda e^{2\pi i\lambda t}$$

We suppose that $\sum|a_\lambda|<\infty$. Then

$$Q_n(x, f)=\sum_{\lambda=1}^{\infty}a_\lambda Q_n(x, e^{2\pi i\lambda t}),$$

where

$$\begin{aligned}Q_n(x, e^{2\pi i\lambda t})&=(n+1)P_n(x, e^{2\pi i\lambda t})-nP_{n-1}(x, e^{2\pi i\lambda t})\\ =\frac{1}{2\pi i\lambda}&[(n+1)^2(1-x(1-e^{2\pi i\lambda/(n+1)}))^n(e^{2\pi i\lambda/(n+1)}-1)\\ &-n^2(1-x(1-e^{2\pi i\lambda/n}))^{n-1}(e^{2\pi i\lambda/n}-1)]\end{aligned}$$

By the mean value theorem, we have for a ξ between n and $n+1$

$$\begin{aligned}2\pi i\lambda Q_n(x, e^{2\pi i\lambda t})&=2\xi(1-x(1-e^{2\pi i\lambda/\xi}))^{\xi-1}(e^{2\pi i\lambda/\xi}-1)\\ &+\xi^2(e^{2\pi i\lambda/\xi}-1)\Big[(1-x(1-e^{2\pi i\lambda/\xi}))^{\xi-1}\log(1-x(1-e^{2\pi i\lambda/\xi}))\\ &\qquad-\frac{2\pi i\lambda x}{\xi}e^{2\pi i\lambda/\xi}(1-x(1-e^{2\pi i\lambda/\xi}))^{\xi-2}\Big]\\ &-\xi^2(1-x(1-e^{2\pi i\lambda/\xi}))^{\xi-1}\frac{2\pi i\lambda}{\xi^2}e^{2\pi i\lambda/\xi}\\ &=Q'+Q''+Q'''\end{aligned}$$

If λ/ξ is sufficiently small, then

$$|Q'|\leqq 4(n+1)\Big(1-4x(1-x)\sin^2\frac{\pi\lambda}{n}\Big)^{n/2}\sin\frac{\pi\lambda}{n}\leqq A\lambda$$

and similarly $|Q'''|\leqq 2\pi\lambda$; and if $\lambda\geqq\xi$ then

$$|Q'|\leqq An,\quad |Q'''|\leqq A\lambda.$$

Hence

$$\sum_{\lambda=1}^{\infty}|a_\lambda|\frac{|Q'|+|Q'''|}{2\pi\lambda}\leqq A\sum_{\lambda=1}^{\infty}|a_\lambda|$$

Let us now estimate Q'' for sufficiently small λ/ξ. We have

*) Here and hereafter we denote by A an absolute constant which is not necessarily the same in each occurrence.

$$Q''=\xi^2(e^{2\pi i\lambda/\xi}-1)\Big[(1-x(1-e^{2\pi i\lambda/\xi}))^{\xi-1}\log(1-x(1-x(1-e^{2\pi i\lambda/\xi})))$$
$$-\frac{2\pi i\lambda x}{\xi}e^{2\pi i\lambda/\xi}(1-x(1-e^{2\pi i\lambda/\xi}))^{\xi-2}\Big]$$
$$=\xi^2\frac{2\pi i\lambda}{\xi}\Big[-(1-x(1-x(1-e^{2\pi i\lambda/\xi})))x(1-e^{2\pi i\lambda/\xi})$$
$$-\frac{2\pi i\lambda x}{\xi}e^{2\pi i\lambda/\xi}\Big](1-x(1-e^{2\pi i\lambda/\xi}))^{\xi-2}(1+o(1))$$
$$=\xi^2\frac{2\pi i\lambda}{\xi}\left(-\frac{x(1+x)}{2}\left(\frac{2\pi i\lambda}{\xi}\right)^2\right)e^{2\pi i\lambda x}(1+o(1))$$
$$=-\frac{x(1+x)}{2}\frac{(2\pi i\lambda)^3}{n}e^{2\pi i\lambda x}(1+o(1))$$

We have also for all λ

$$|Q''|\leqq A(n^2+\lambda n)$$

Therefore, if there is an infinitude of n such that a_λ vanishes except for $\lambda\leqq n$ with sufficiently small λ/n, then we have for every such n

$$Q_n(x,f)=2\pi^2\frac{x(1+x)}{n}\sum_{\lambda=1}^{n}\lambda^2 a_\lambda(1+o(1))e^{2\pi i\lambda x}+\theta_n n\sum_{\lambda=n+1}|a_\lambda|+\tau_n,$$

where θ_n and τ_n are bounded.

Thus, in order to prove the theorem, it suffices to take a_λ for which there is a sequence (n_k) satisfying the above condition, such that

$$\frac{1}{n_k}\sum_{\lambda=1}^{n_k}\lambda^2 a_\lambda(1+o(1))e^{2\pi i\lambda x}$$

diverges to infinity almost everywhere and

$$n_k\sum_{\lambda=n_k+1}^{\infty}|a_\lambda|=O(1).$$

For example, we may take

$$n_1=2^2,\quad n_{k+1}=2^{2^{n_k}}\quad(k\geqq 2)$$

and

$$a_\lambda=\frac{1}{\nu^2}\quad\text{for } \lambda=2^\nu,\ n_k\leqq\lambda\leqq kn_k\quad(k=1,2,\cdots),$$
$$=0\quad\text{otherwise.}$$

References

[1] G. G. Lorentz: Bernstein Polynomials, Toronto (1953).
[2] P. L. Butzer: Summability of generalized Bernstein polynomials. I, Duke Math. Jour., **22**, 617-627 (1955).