75. Fourier Series. XVII. Order of Partial Sums and Convergence Theorem

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1. **Introduction.** Let $f(t)$ be an integrable function with period 2π and its Fourier series be

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt). \tag{1}$$

By $s_n(x)$ we denote the nth partial sum of the Fourier series (1). We put as usual $\varphi_x(t)=f(x+t)+f(x-t)$.

We have proved the following theorems in [1].

Theorem 1. *If*

$$\int_0^t \varphi_x(u)\,du=o(t) \qquad (t\to 0) \tag{2}$$

and

$$\int_0^t (f(\xi+u)-f(\xi-u))\,du=o(t) \qquad (t\to 0) \tag{3}$$

uniformly in ξ *in a neighbourhood of* x, *then*

$$s_n(x)=o(\log n).$$

Theorem 2. *If*

$$\int_0^t \varphi_x(u)\,du=o\left(t\Big/\log\frac{1}{t}\right) \qquad (t\to 0) \tag{4}$$

and

$$\int_0^t (f(\xi+u)-f(\xi-u))\,du=o\left(t\log\log\frac{1}{t}\Big/\log\frac{1}{t}\right) \qquad (t\to 0) \tag{5}$$

uniformly in ξ *in a neighbourhood of* x, *then*

$$s_n(x)=o(\log\log n).$$

By the same way as the proof of these theorems, we get the following generalizations.

Theorem 3. *Let* $0\leqq\alpha\leqq 1$. *If*

$$\int_0^t \varphi_x(u)\,du=o\left(t\left(\log\frac{1}{t}\right)^{\alpha}\right) \qquad (t\to 0) \tag{6}$$

and

$$\int_0^t (f(\xi+u)-f(\xi-u))\,du=o\left(t\Big/\left(\log\frac{1}{t}\right)^{1-\alpha}\right) \qquad (t\to 0) \tag{7}$$

uniformly in ξ *in a neighbourhood of* x, *then*

$$s_n(x)=o((\log n)^{\alpha}). \tag{8}$$

Theorem 4. *Let* $0\leqq\alpha\leqq 1$. *If*

(9) $$\int_0^t \varphi_x(u)\,du = o\left(t\left(\log\log\frac{1}{t}\right)^{\alpha}\right) \quad (t\to 0)$$

and

(10) $$\int_0^t (f(\xi+u)-f(\xi-u))\,du = o\left(t\left(\log\log\frac{1}{t}\right)^{\alpha}\Big/\log\frac{1}{t}\right) \quad (t\to 0)$$

uniformly in ξ in a neighbourhood of x, then

(11) $$s_n(x) = o\left((\log\log n)^{\alpha}\right).$$

In the case $\alpha=1$, the conditions (6) and (9) in Theorems 3 and 4 are more general than (2) and (4) in Theorems 1 and 2, respectively. The case $\alpha=0$ becomes a convergence theorem proved in [2].

In § 2 we prove Theorem 3; since Theorem 4 may be quite similarly proved we shall omit its proof.

We can generalize the last two theorems into the following form.

Theorem 5. *Let $0\leqq\alpha\leqq 1$. If*

(12) $$\frac{1}{t}\int_0^t (t-u)\varphi_x(u)\,du = o\left(t\left(\log\frac{1}{t}\right)^{\alpha}\right) \quad (t\to 0)$$

and

(13) $$\frac{1}{t}\int_0^t (t-u)(f(\xi+u)-f(\xi-u))\,du = o\left(t\Big/\left(\log\frac{1}{t}\right)^{1-\alpha}\right) \quad (t\to 0)$$

uniformly in ξ in a neighbourhood of x, then (8) holds.

Theorem 6. *Let $0\leqq\alpha\leqq 1$. If*

(14) $$\frac{1}{t}\int_0^t (t-u)\varphi_x(u)\,du = o\left(t\left(\log\log\frac{1}{t}\right)^{\alpha}\right) \quad (t\to 0)$$

and

(15) $$\frac{1}{t}\int_0^t (t-u)(f(\xi+u)-f(\xi-u))\,du = o\left(t\left(\log\log\frac{1}{t}\right)^{\alpha}\Big/\log\frac{1}{t}\right) \quad (t\to 0)$$

uniformly in ξ in a neighbourhood of x, then (11) holds.

The left side terms of (12) - (15) are the (C, 1) mean of those of (6), (7), (9), (10), respectively.

In § 3 we prove Theorem 5. Theorem 6 may be proved similarly to Theorem 5, so that its proof will be left for readers.

As an application of Theorems 5 and 6, we get the following theorems.

Theorem 7. *If the condition (13) holds, then*

(16) $$\frac{a_0}{2}+\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{1+\varepsilon(\log n)^{\alpha}} \to f(x) \quad (\varepsilon\to 0)$$

at the Lebesgue point x.

Theorem 8. *If the condition (15) holds, then*

(17) $$\frac{a_0}{2}+\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{1+\varepsilon(\log\log n)^{\alpha}} \to f(x) \quad (\varepsilon\to 0)$$

at the Lebesgue point x.

The summability of the type (16) ($\alpha=1$) was first introduced by R. Salem [3] (cf. [4]). The general case is considered by M. Kinukawa.*) Proof of these theorems follows from Theorem 5 and Theorem 6 and the method used by S. Izumi and T. Kawata [4] (cf. [5]), so that we shall omit it.

2. Proof of Theorem 3. We shall sketch the proof of Theorem 3. We write

$$s_n(x)=\frac{1}{\pi}\int_0^{\pi}\varphi_x(t)\frac{\sin nt}{t}dt+o(1)$$
$$=\frac{1}{\pi}\Big[\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\Big]+o(1)=\frac{1}{\pi}[I+J]+o(1).$$

Let $\Phi_x(t)=\int_0^t\varphi_x(u)\,du$, then integrating by parts and using the condition (6) we get

$$|I|\leqq\int_0^{\pi/n}|\Phi_x(t)|\left|\frac{\sin nt}{t^2}-\frac{n\cos nt}{t}\right|dt=o\Big(n\int_0^{\pi/n}\Big(\log\frac{1}{t}\Big)^{\alpha}dt\Big)$$
$$=o\,((\log n)^{\alpha}).$$

We write as in [1]

$$J=J_1+J_2+o(1),$$

where

$$J_1=\sum_{k=1}^{[(n-1)/2]}\int_0^{\pi/n}\frac{\varphi_x(t+2k\pi/n)-\varphi_x(t+(2k-1)\pi/n)}{t+2k\pi/n}\sin nt\,dt,$$
$$J_2=\sum_{k=1}^{[(n-1)/2]}\int_0^{\pi/n}\varphi_x(t+(2k-1)\pi/n)\Big(\frac{1}{t+2k\pi/n}-\frac{1}{t+(2k-1)\pi/n}\Big)\sin nt\,dt.$$

Then by the second mean value theorem, for $0\leqq\eta_k<\xi_k\leqq\pi/n$,

$$J_1=\sum_{k=1}^{[(n-1)/2]}\frac{n}{2k\pi}\int_{\eta_k}^{\xi_k}\{\varphi_x(t+2k\pi/n)-\varphi_x(t+(2k-1)\pi/n)\}\,dt$$
$$=o\Big(n\sum_{k=1}^{n}\frac{1}{k}\frac{1}{n\,(\log n)^{1-\alpha}}\Big)=o\,((\log n)^{\alpha}),$$

using the condition (7).

On the other hand by Abel's lemma we write

$$J_2=\sum_{k=1}^{[(n-1)/2]}\int_0^{\pi/n}\sum_{j=k}^{n}\Big(\frac{1}{t+2j\pi/n}-\frac{1}{t+(2j-1)\pi/n}\Big)$$
$$(\varphi_x(t+(2k-1)\pi/n)-\varphi_x(t+(2k-3)\pi/n))\sin nt\,dt$$
$$+\int_0^{\pi/n}\sum_{j=1}^{n}\Big(\frac{1}{t+2j\pi/n}-\frac{1}{t+(2j-1)\pi/n}\Big)\varphi_x(t+\pi/n)\sin nt\,dt$$
$$=J_{21}+J_{22},$$

then by the second mean value theorem and condition (7) we have, for $0\leqq\eta_k'<\xi_k'\leqq\pi/n$,

*) His result is not published.

$$J_{21}=\frac{\pi}{n}\sum_{k=1}^{[(n-1)/2]}\sum_{j=k}^{n}\frac{n^2}{(2j-1)^2\pi^2}\int_{\eta_k'}^{\xi_k'}\{\varphi_x(t+(2k-1)\pi/n)-\varphi_x(t+(2k-3)\pi/n)\}\,dt$$

$$=o\Big(\sum_{k=1}^{n}\sum_{j=k}^{n}\frac{n}{j^2}\,\frac{1}{n(\log n)^{1-\alpha}}\Big)=o\Big(\sum_{k=1}^{n}\frac{1}{k}\,\frac{1}{(\log n)^{1-\alpha}}\Big)=o\,((\log n)^{\alpha}),$$

and by the second mean value theorem and condition (6) we have, for $0\leqq\eta_j'<\xi_j'\leqq\pi/n$,

$$J_{22}=\frac{\pi}{n}\sum_{j=1}^{n}\frac{n^2}{(2j-1)^2\pi^2}\int_{\eta_j'}^{\xi_j'}\varphi_x(t+\pi/n)\,dt=o\Big(n\sum_{j=1}^{n}\frac{1}{j^2}\,\frac{(\log n)^{\alpha}}{n}\Big)$$

$$=o\,((\log n)^{\alpha}).$$

Collecting above estimations we get the required.

3. **Proof of Theorem 5.** We have by integration by parts

$$s_n(x)=\frac{1}{\pi}\int_0^{\pi}\varphi_x(t)\frac{\sin nt}{t}\,dt+o(1)$$

$$=\frac{1}{\pi}\Big[-n\int_0^{\pi}\Phi_x(t)\frac{\cos nt}{t}\,dt+\int_0^{\pi}\Phi_x(t)\frac{\sin nt}{t^2}\,dt\Big]+o(1)$$

$$=\frac{1}{\pi}[-I+J]+o(1).$$

In order to estimate I, we divide it such that

$$I=n\Big[\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\Big]\Phi_x(t)\frac{\cos nt}{t}\,dt=I_1+I_2.$$

Integrating by parts and using the condition (12) we get

$$I_1=n\int_0^{\pi/n}\Phi_x(t)\frac{\cos nt}{t}\,dt$$

$$=\Big[n\Phi_x^*(t)\frac{\cos nt}{t}\Big]_0^{\pi/n}+n\int_0^{\pi/n}\Phi_x^*(t)\Big[\frac{n\sin nt}{t}+\frac{\cos nt}{t^2}\Big]dt$$

$$=o\Big(\Big[nt\Big(\log\frac{1}{t}\Big)^{\alpha}\Big]_0^{\pi/n}\Big)+o\Big(n^3\int_0^{\pi/n}t^2\Big(\log\frac{1}{t}\Big)^{\alpha}dt+n\int_0^{\pi/n}\Big(\log\frac{1}{t}\Big)^{\alpha}dt\Big)$$

$$=o\,((\log n)^{\alpha}),$$

where $\Phi_x^*(t)=\int_0^t\Phi_x(u)\,du$.

On the other hand we write, $N=[(n-1)/2]$,

$$I_2=n\int_{\pi/n}^{\pi}\Phi_x(t)\frac{\cos nt}{t}\,dt=n\sum_{k=1}^{n-1}\int_{k\pi/n}^{(k+1)\pi/n}\Phi_x(t)\frac{\cos nt}{t}\,dt$$

$$=n\sum_{k=1}^{n-1}(-1)^k\int_0^{\pi/n}\Phi_x(t+k\pi/n)\frac{\cos nt}{t+k\pi/n}\,dt$$

$$=n\sum_{k=1}^{N}\int_0^{\pi/n}\Big\{\frac{\Phi_x(t+2k\pi/n)}{t+2k\pi/n}-\frac{\Phi_x(t+(2k-1)\pi/n)}{t+(2k-1)\pi/n}\Big\}\cos nt\,dt+o(1)$$

$$=n\sum_{k=1}^{N}\int_0^{\pi/n}\frac{\Phi_x(t+2k\pi/n)-\Phi_x(t+(2k-1)\pi/n)}{t+2k\pi/n}\cos nt\,dt$$

$$+n\sum_{k=1}^{N}\int_0^{\pi/n}\Phi_x(t+(2k-1)\pi/n)\Big(\frac{1}{t+2k\pi/n}-\frac{1}{t+(2k-1)\pi/n}\Big)\cos nt\,dt+o(1)$$
$$=I_{21}+I_{22}+o(1).$$

Now we have

$$I_{21}=n\sum_{k=1}^{N}\Big[\int_0^{\pi/2n}+\int_{\pi/2n}^{\pi/n}\Big]\frac{\Phi_x(t+2k\pi/n)-\Phi_x(t+(2k-1)\pi/n)}{t+2k\pi/n}\cos nt\,dt$$
$$=n\sum_{k=1}^{N}\int_0^{\pi/2n}\Big\{\frac{\Phi_x(t+2k\pi/n)-\Phi_x(t+(2k-1)\pi/n)}{t+2k\pi/n}$$
$$-\frac{\Phi_x((2k+1)\pi/n-t)-\Phi_x(2k\pi/n-t)}{(2k+1)\pi/n-t}\Big\}\cos nt\,dt$$
$$=n\sum_{k=1}^{N}\int_0^{\pi/2n}\{\Phi_x(t+2k\pi/n)-\Phi_x(t+(2k-1)\pi/n)$$
$$-\Phi_x((2k+1)\pi/n-t)+\Phi_x(2k\pi/n-t)\}\frac{\cos nt}{t+2k\pi/n}\,dt$$
$$+n\sum_{k=1}^{N}\int_0^{\pi/2n}\{\Phi_x((2k+1)\pi/n-t)-\Phi_x(2k\pi/n-t)\}$$
$$\Big\{\frac{1}{2k\pi/n+t}-\frac{1}{(2k+1)\pi/n-t}\Big\}\cos nt\,dt$$
$$=I_{211}+I_{212}.$$

Concerning I_{211} we get, for $0\leqq\xi_k<\pi/2n$,

$$I_{211}=n\sum_{k=1}^{N}\int_0^{\pi/2n}\frac{\cos nt}{2k\pi/n+t}dt\Big\{\int_{(2k-1)\pi/n+t}^{2k\pi/n+t}\varphi_x(u)\,du-\int_{2k\pi/n-t}^{(2k+1)\pi/n-t}\varphi_x(u)\,du\Big\}$$
$$=n\sum_{k=1}^{N}\frac{n}{2k\pi}\int_{\xi_k}^{\pi/2n}dt\Big\{\int_{(2k-1)\pi/n+t}^{2k\pi/n-t}\varphi_x(u)\,du-\int_{2k\pi/n+t}^{(2k+1)\pi/n-t}\varphi_x(u)\,du\Big\}$$
$$=n^2\sum_{k=1}^{N}\frac{1}{2k\pi}\int_{\xi_k}^{\pi/2n}dt\int_t^{\pi/n-t}[\varphi_x(2k\pi/n-u)-\varphi_x(2k\pi/n+u)]du$$
$$=\frac{n^2}{2\pi}\sum_{k=1}^{N}\frac{1}{k}\Big[\int_{\pi/2n}^{\pi/n-\xi_k}-\int_{\xi_k}^{\pi/2n}\Big]dt\int_0^t[\varphi_x(2k\pi/n-u)-\varphi_x(2k\pi/n+u)]\,du$$
$$=o\Big(n^2\sum_{k=1}^{N}\frac{1}{k}\,\frac{1}{n^2}(\log n)^{\alpha-1}\Big)=o\,((\log n)^{\alpha}),$$

by the condition (13). On the other hand

$$I_{212}=n\sum_{k=1}^{N}\int_0^{\pi/2n}\sum_{j=k}^{n}\Big\{\frac{1}{2j\pi/n+t}-\frac{1}{(2j+1)\pi/n-t}\Big\}\cos nt$$
$$\{\Phi_x((2k+1)\pi/n-t)-\Phi_x(2k\pi/n-t)-\Phi_x((2k-1)\pi/n-t)$$
$$+\Phi_x((2k-2)\pi/n-t)\}\,dt$$
$$+n\int_0^{\pi/2n}\sum_{j=1}^{n}\Big\{\frac{1}{2j\pi/n+t}-\frac{1}{(2j+1)\pi/n-t}\Big\}\{\Phi_x(3\pi/n-t)-\Phi_x(2\pi/n-t)\}$$
$$\cos nt\,dt$$
$$=I_{2121}+I_{2122}.$$

Since $(2j\pi/n+t)^{-1}-((2j+1)\pi/n-t)^{-1}$ is positive and decreasing, and its maximum value is $n^2/2j(2j+1)\pi^2$ for $0\leqq t\leqq\pi/2n$. Hence, for $0\leqq\xi_k<\eta_k\leqq\pi/2n$,

$$I_{2121}=\frac{n^2}{\pi^2}\sum_{k=1}^{N}\Big(\sum_{j=k}^{n}\frac{1}{2j(2j+1)}\Big)\int_{\xi_k}^{\eta_k}dt\int_t^{\pi/n-t}[\varphi_x(2k\pi/n-u)-\varphi_x(2k\pi/n+u)]du$$
$$=o\Big(n^2\sum_{k=1}^{N}\frac{1}{k}\frac{(\log n)^{\alpha-1}}{n^2}\Big)=o((\log n)^\alpha),$$

by the condition (13). Similarly, for $0\leqq\xi_j<\pi/2n$,

$$I_{2122}=\frac{n^2}{\pi^2}\sum_{j=1}^{n}\frac{1}{2j(2j+1)}\int_{\xi_j}^{\pi/2n}\{\Phi_x(3\pi/n-t)-\Phi_x(2\pi/n-t)\}\,dt$$
$$=O\Big(n^2\int_{\xi_j}^{\pi/2n}\int_{2\pi/n-t}^{3\pi/n-t}\varphi_x(u)\,du\Big)=o\Big(n^2\frac{(\log n)^\alpha}{n^2}\Big)=o((\log n)^\alpha),$$

by the condition (12). Collecting above estimations, we get $I=o((\log n)^\alpha)$.

On the other hand we set

$$J=\Big[\int_0^{\pi/2n}+\int_{\pi/2n}^{\pi}\Big]\Phi_x(t)\frac{\sin nt}{t^2}dt=J_1+J_2,$$

then

$$J_1=\int_0^{\pi/2n}\Phi_x(t)\frac{\sin nt}{t^2}dt$$
$$=\Big[\Phi_x^*(t)\frac{\sin nt}{t^2}\Big]_0^{\pi/2n}+\int_0^{\pi/2n}\Phi_x^*(t)\Big(\frac{2\sin nt}{t^3}-\frac{n\cos nt}{t^2}\Big)dt$$
$$=o((\log n)^\alpha)+o(1)+o\Big(n\int_0^{\pi/2n}\Big(\log\frac{1}{t}\Big)^\alpha dt\Big)=o((\log n)^\alpha)$$

and

$$J_2=\sum_{k=1}^{n-1}\int_{k\pi/n-\pi/2n}^{(k+1)\pi/n-\pi/2n}\Phi_x(t)\frac{\sin nt}{t^2}dt$$
$$=\sum_{k=1}^{n-1}\int_0^{\pi/n}(-1)^k\Phi_x(t+(k-1/2)\pi/n)\frac{\cos nt}{(t+(k-1/2)\pi/n)^2}dt$$
$$=\sum_{k=1}^{n-1}\int_0^{\pi/n}\Big\{\frac{\Phi_x(t+(2k-1/2)\pi/n)}{(t+(2k-1/2)\pi/n)^2}-\frac{\Phi_x(t+(2k-3/2)\pi/n)}{(t+(2k-3/2)\pi/n)^2}\Big\}\cos nt\,dt+o(1).$$

The estimation of the last sum runs similarly as I_2, so that we shall omit it. Thus we have proved the theorem completely.

Finally the author expresses her hearty thanks to Mr. M. Kinukawa who showed her his unpublished paper proving that (8) and (11) hold almost everywhere under the condition (7) for $0<\alpha<1$. Above method of proof is quite different from his.

References

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