91. Notes on Knots and Periodic Transformations

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Introduction. Let T be a sense preserving periodic transformation of the 3-sphere S^3 onto itself. Furthermore let T be different from the identity and have at least one fixed point. Then it has been shown by Smith[9] that the set F of all fixed points of T is a simple closed curve. Recently Montgomery, Zippin and Samelson[5][6] have studied about the position of F in S^3, which also be concerned in this note. Hereafter we always assume that T is semilinear, and then F is polygonal. Let p be the period of T. Identifying the points

$$x,\ T(x),\cdots,\ T^{p-1}(x)$$

in S^3, we have an orientable 3-manifold M. Then it will be proved in § 4 that M is simply connected, i.e. the fundamental group of M consists of only one element. In § 5, under the assumption that the well-known Poincaré conjecture on 3-manifolds is true, we shall prove that almost all knots of the Alexander-Briggs's table[1] are not equivalent to F, if T is of period 2. This will be done by the use of Alexander polynomials.[2] To prove these we shall study knots in 3-manifolds in §§ 1–3. *In this note everything will be considered from the semilinear point of view.*

§ 1. Let M be a compact 3-manifold (without boundary) and k an oriented simple closed curve in M. The fundamental group of $M-k$ will be denoted by $F(M-k)$ or sometimes by $F(k, M)$. *Hereafter we always assume that k is homologous to* 0 *in M.* Let V be a sufficiently small tubular neighbourhood of k in M. Then the boundary $\dot{V}$ of V is a torus. A meridian of $\dot{V}$ is a simple closed curve on $\dot{V}$ which bounds a 2-cell in V but not on $\dot{V}$. Let x be an oriented meridian of $\dot{V}$. Since k is homologous to 0 in M, the linking number Link(k,x) of k and x can be defined and is equal to ± 1. We may always suppose that x is oriented such that

$$\text{Link}\,(k, x) = 1.$$

For each integer $p(\neq 0)$ x^p is not homotopic to 1.

Now let $\{x, X_1, X_2,\cdots, X_n\}$ be the set of generators of $F(M-k)$. Put

$$\text{Link}\,(k, X_i)=v(i) \qquad (i=1, 2,\cdots, n)$$

and

$$x_i=x^{-v(i)}X_i. \qquad (i=1, 2,\cdots, n)$$

Then $\{x, x_1, x_2, \cdots, x_n\}$ forms again the set of generators of $F(M-k)$ and for each i

(1) $$\text{Link}(k, x_i) = 0.$$

Let $R_s=1$ $(s=1, 2, \cdots, m)$ be defining relations of $F(M-k)$ with respect to $\{x, x_i\}$. Then the symbol

(2) $$\{x, x_1, \cdots, x_n : R_1, \cdots, R_m\}$$

will be called a *presentation*[3] of $F(M-k)$. A presentation of $F(M)$ is given by

(3) $$\{x, x_1, \cdots, x_n : x, R_1, \cdots, R_m\}.$$

§ 2. Let $w \in F(k, M)$. Then w is written as a word which consists of at most $x, x_1, \cdots, x_n$. Let $f(w)$ be an integer which is equal to the exponent sum of w, summed over the element x. By (1) it is easy to see that f is a homomorphism of $F(k, M)$ onto the set of all integers. Now put

$$F_g(k, M) = \{w \in F(k, M) \mid f(w) = 0 \pmod{g}\},$$

where $g>0$. Then $F_g(k, M)$ is a normal subgroup of $F(k, M)$. Therefore there exists uniquely the *g-fold cyclic covering space* $\tilde{M}_g(k)$[7] of $M-k$, whose fundamental group is isomorphic to $F_g(k, M)$. Since x is a meridian of $\dot{V}$, we can also define the *g-fold cyclic covering space* $M_g(k)$ of M, branched along k. For each g $M_g(k)$ is a closed 3-manifold.

$F(\tilde{M}_g(k))$ and $F(M_g(k))$ are calculated from $F(k, M)$ as follows: Let (2) be a presentation of $F(k, M)$. Put

$$x_{ij} = x^j x_i x^{-j}. \quad \begin{pmatrix} i=1, 2, \cdots, n \\ j=0, 1, \cdots, g-1 \end{pmatrix}$$

Since $f(R_s)=0$ for every s $(s=1, 2, \cdots, m)$, $x^j R_s x^{-j}$ is expressible by a word which consists of at most x_{ij} and x^g. We denote it by notations

$$x^j R_s x^{-j} = \tilde{R}_s.$$

Then

(4) $$\{x^g, x_{ij} : \tilde{R}_s\}$$

is a presentation of $F(\tilde{M}_g(k))$ and

(5) $$\{x^g, x_{ij} : x^g, \tilde{R}_s\}$$

is one of $F(M_g(k))$.

There is a homomorphism of $F(M_g(k))$ onto $F(M)$. To prove this: let (3) and (5) be presentations of $F(M)$ and $F(M_g(k))$, respectively. Put $h(x_{ij})=x_i$ for each x_{ij}. It is easy to see that h can be extended to a homomorphism of $F(M_g(k))$ onto $F(M)$.

From the above fact we have immediately the following

Theorem 1. *Let M be a 3-manifold and k a simple closed curve in M which is homologous to 0 in M. If a g-fold cyclic covering space $M_g(k)$ of M, branched along k, is simply connected, then M is also simply connected.*

§ 3. Let (2) be a presentation of $F(k, M)$. Put

$$x^j x_i^{\pm 1} x^{-j} = \pm x^j x_i \quad \begin{pmatrix} i=1, 2, \cdots, n \\ j=0, \pm 1, \pm 2, \cdots \end{pmatrix}$$

and replace the multiplication by the addition. Furthermore suppose that the addition is commutative. Then for each relation $R_s=1$ $(s=1, 2, \cdots, m)$ we have a relation $\overline{R}_s=0$, which is a linear equation of x_i. If $m<n$, then we add $n-m$ trivial equations $0=0$ to the system of equations and then we may assume that $m \geqq n$. From these linear equations we can make the Alexander matrix, whose (s, i)-th term is the coefficient of x_i in $\overline{R}_s=0$.

If two oriented knots k_1 and k_2 in M are equivalent each other, then $F(k_1, M)$ and $F(k_2, M)$ are *directly isomorphic.*[2] It was proved by Alexander[2] that if two indexed groups are directly isomorphic each other, then the elementary factors different from unity of the Alexander matrices and also their products $\Delta(x, k, M)$ are the same each other. Of course they are determined up to factors $\pm x^p$. It should be remarked that $\Delta(x, k, M_g(k))$ is also defined from (4) replacing x^g by x.

It can be proved that

$$\Delta(x, k, M_g(k)) = \Pi_{j=0}^{g-1} \Delta(\sqrt[g]{x}\, \omega_j, k, M), \tag{6}$$

where $\omega_j = \cos\dfrac{2\pi j}{g} + i \sin\dfrac{2\pi j}{g}$. This is known for the case $M=S^3$.[4] But the proof of the latter depends essentially only on the following equation of determinants:

$$\begin{vmatrix} a_1 & a_2 \cdots & a_g \\ xa_g & a_1 \cdots & a_{g-1} \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ xa_2 & xa_3 \cdots & a_1 \end{vmatrix} = \Pi_{j=0}^{g-1} f(\sqrt[g]{x}\, \omega_j),$$

where $f(y) = a_1 + a_2 y + \cdots + a_g y^{g-1}$. Therefore the proof of our case is the same as the case $M=S^3$ and is omitted. As a special case of (6) we have

$$\Delta(1, k, M_g(k)) = \Pi_{j=0}^{g-1} \Delta(\omega_j, k, M).$$

$\Delta(1, k, M_g(k)) \neq 0$ if and only if the 1-dimensional Betti number $p_1(M_g(k)) = 0$. If $p_1(M_g(k))=0$, then $|\Delta(1, k, M_g(k))|$ is equal to the product of torsion numbers (in this case if $|\Delta(1, k, M_g(k))|=1$, then $M_g(k)$ has no torsion number).

§ 4. Now let T be a sense preserving (of course semilinear) periodic transformation of S^3 onto itself. Furthermore let T be different from the identity and have at least one fixed point. Then the set F of all fixed points of T is a simple closed curve.[9] Suppose that p is the minimal number of the set of all positive periods of T.

It is easy to see that T is *primitive*.[10] Identifying the points

$$x,\ T(x),\cdots,\ T^{p-1}(x)$$

in S^3, we have an orientable 3-manifold M. *Hereafter we always use M only in this meaning.* Since F is homologous to 0 in S^3, F is homologous to 0 in M. T acts locally as a rotation about F.[5] From this it follows that *S^3 is the p-fold cyclic covering space of M, branched along k.* Then by Theorem 1 we have the following

Theorem 2. *Suppose that T and M have the above meaning. Then M is simply connected.*

§ 5. Now we assume that the Poincaré conjecture is true. Then by Theorem 2 M is a 3-sphere, and the coefficients of $\Delta(x, F, M)$ are symmetric.[8][11] We consider in § 5 only the case $p=2$.

Suppose first that the degree of $\Delta(x, F, S^3)$ is 2. Then by (6) the degree of $\Delta(x, F, M)$ is also 2. Put

$$\Delta(x, F, M)=ax^2+bx+a,$$

where $a\neq 0$ and we may assume that $2a+b=1$. Then by (6)

$$\Delta(x, F, S^3)=a^2x^2+(2a^2-b^2)x+a^2.$$

Furthermore $4a^2-b^2=\pm 1$, which means that $2a-b=\pm 1$. From this it follows that $2a=1$ or $2a=0$. Since $a\neq 0$ and a is an integer, this is a contradiction. Thus we have proved that *if the degree of $\Delta(x, k, S^3)$ is 2, then k is not equivalent to F.*

By the same way it can be seen easily that if the degrees of $\Delta(x, F, S^3)$ are 4, 6 and 8, then $\Delta(x, F, S^3)$ are confined to the following forms, respectively:

$$a^2x^4+2a(1-2a)x^3+(1-4a+6a^2)x^2+\cdots,$$

$$a^2x^6-(2a^2+b^2)x^5-(a^2+2b-4b^2)x^4 +(4a^2-1+4b-6b^2)x^3-\cdots,$$

$$a^2x^8+(2ac-b^2)x^7+(2a-4ac-4a^2+c^2+2b^2)x^6 +(2c-2ac-4c^2+b^2)x^5+(1-4a-4c+8ac +6a^2+6c^2-4b^2)x^4+\cdots\ .$$

From this we have the following

Theorem 3. *Let T be the same as that of Theorem 2. Furthermore suppose that the period of T is 2. Then, under the assumption that the Poincaré conjecture is true, all knots of the Alexander-Briggs's table,*[1][2] *except for the cases 8_9 and 8_{20}, are not equivalent to F.*

Remark. If we do not assume that the Poincaré conjecture is true, then we have the following exceptional case:

$$\Delta(x, F, S^3)=a^2x^2-(2a^2+1)x+a^2,$$

even if the degree of $\Delta(x, F, S^3)$ is 2. The exceptional cases of higher degrees will be more complicated.

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