113. *The Initial Value Problem for Linear Partial Differential Equations with Variable Coefficients. III*

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In the present paper we consider mixed problems of linear parabolic equations with boundary conditions formulated by J. L. Lions [4] such that, using his notations, V is independent of the time variables, but N_t depends on them.

Our methods (§2), are also applicable to mixed problem of linear equations of many other types with above-mentioned boundary conditions, with which it seems interesting to me to compare Kato's methods [2].

As an illustlation of our considerations we consider in §3 the Fokker-Planck's equations formulated by K. Yosida [7].

Only a sketch of this proof will be given, however, the details with further investigations will be published elsewhere.

1. Preliminary. Let Ω be a domain of the Euclidean space. Let $((u, v))_t$ be real bilinear forms defined on a real separable Hilbert space V with following conditions, where $\mathfrak{D}(\Omega)\subset V\subset L_2(\Omega)$ and the injections $\mathfrak{D}(\Omega)\to V$, $V\to L_2(\Omega)$ are both continuous: there are positive constants a, b, c such that for any $t(-\infty<t<\infty)$

(I)
$$((u, u))_t \geqq a\,\|u\|_V^2$$
$$b\,\|u\|_V\,\|v\|_V \geqq |((u, v))_t|,$$

(II) for fixed u, $v\in V$,
$$c\,|t-t'|\,\|u\|_V\,\|v\|_V \geqq |((u, v))_t-((u, v))_{t'}|.$$

Furthermore let $\bar{A}_t$ be an operator in $L_2(\Omega)$ into itself such that $f\in D(\bar{A}_t)$ if and only if $((f, v))_t=(\bar{A}_t f, v)_{L_2(\Omega)}$ for every $v\in V$, where $(\bar{A}_t f, v)_{L_2(\Omega)}=A_t f(v)$ for the distribution $A_t f$ defined by the relation: $((f, v))_t=A_t f(v)$ for every $v\in\mathfrak{D}(\Omega)$. Then $\bar{A}_t$ is a densely defined, closed operator in $L_2(\Omega)$ into itself whose adjoint coincides with operator $\bar{A}_t^*$ defined as above from $((u, v))_t^*=((v, u))_t$ [3, 4]. Let G_t^* be the Green operators with respect to the form $((u, v))_t^*$. Then from (II) we see the following

Lemma 1. *For any $u\in L_2(\Omega)$, $G_t^* u$ is differentiable in V (a.e.t) and $\frac{d}{dt}G_t u$ is measurable with respect to V such that*

$$\left\|\frac{d}{dt}G_t u\right\|_V \leqq c\,\|u\|_V \qquad (a.e.t).$$

Definition. Let E be a real separable Hilbert space. Then we denote by $\mathfrak{L}^n(E)$ the completion of the real linear space $\mathfrak{D}_t(E)$ with the following norm:

$$\int_{-\infty}^{\infty}((q^n u(t),\ u(t)))_E\,dt=\int_{-\infty}^{\infty}(1+|\xi|^2)^n\,\|\mathfrak{F}(u)(\xi)\|_E^2\,d\xi,$$

where $q=\alpha-\dfrac{d^2}{dt^2}\,(\alpha>0)$, $\mathfrak{F}(u)$ is the Fourier transform of u and where $\|\mathfrak{F}(u)(\xi)\|_E$ is the complex Hilbert space-norm extended from the real norm of E.

Lemma 2. *For any integers s, m, q^s is an isomorphism from $\mathfrak{L}^{m+s}(E)$ onto $\mathfrak{L}^{m-s}(E)$. Furthermore $\dfrac{d}{dt}$ is a continuous operator from $\mathfrak{L}^m(E)$ into $\mathfrak{L}^{m-1}(E)$.*

2. Parabolic equations. Let $(((u, v)))_t$ be a real bilinear form defined on V with the following properties: there are positive constants α such that the bilinear form

$$(((u, v)))_t+\alpha(u, v)_{L_2(\Omega)}$$

satisfies the conditions (I), (II) and in §1. Then for sufficiently large α and β, $q=\beta-\dfrac{d}{dt^2}$,

$$\int_{-\infty}^{\infty}(((qu, u)))_t\,dt\geqq\gamma\int_{-\infty}^{\infty}(qu, u)_{L_2(\Omega)}\,dt\quad(\gamma>0)$$

where $u=u(t)\in\mathfrak{D}_t(V)$. Therefore setting

$$((u, v))_t=(((u, v)))_t+\alpha(u, v)_{L_2(\Omega)}$$

we see the following

Lemma 3. *For any $u(t)\in q^{-1}G_t^*(\mathfrak{D}((-\infty,\infty)\times\Omega))$ and for sufficiently large β, and for some $\gamma(\beta)>0$*

$$\int_{-\infty}^{\infty}\int_{\Omega}\Big(-\frac{d}{dt}+\bar{A}_t^*\Big)q\ u(t)\cdot u(t)\,dx\,dt\geqq\gamma(\beta)\int_{-\infty}^{\infty}\int_{\Omega}qu\cdot u\,dx\,dt.$$

From Lemma 3 it follows that for any $u(t)\in q^{-1}G_t^*(\mathfrak{D}(-\infty,\infty)\times\Omega))$

$$\Big\|q^{-1}\Big(-\frac{d}{dt}+\bar{A}_t^*\Big)q\ u\Big\|_{\mathfrak{L}^1(L_2(\Omega))}\geqq\gamma'\,\|u\|_{\mathfrak{L}^1(L_2(\Omega))}\quad(\gamma'>0).$$

Therefore by a limit process we see the following

Theorem 1. *For any $g(t)\in\mathfrak{L}^1(L_2(\Omega))$ there is a solution $f(t)\in\mathfrak{L}^1(L_2(\Omega))\frown\mathfrak{L}^0(V)$ such that*

$$f(t)\in D(\bar{A}_t)\quad(a.e.t)$$

$$\Big(\frac{d}{dt}+\bar{A}_t\Big)f(t)=g(t)\qquad(a.e.t).$$

Furthermore such a solution $f(t)$ satisfies the following inequality:

$$\Big\|\Big(\frac{d}{dt}+\bar{A}_t\Big)f\Big\|_{\mathfrak{L}^0(L_2(\Omega))}\geqq\gamma''\,\|f\|_{\mathfrak{L}^0(L_2(\Omega))}\quad(\gamma''>0),$$

which implies the uniqueness of solution in Theorem 1. Moreover, since $\mathfrak{L}^1(L_2(\Omega))\subset C_t(L_2(\Omega))$, we see that for such a solution $f(t)$, and for b, a $(b>a)$

$$\| f(b) \|^2_{L_2(\Omega)} \leqq 2\left\{\int_a^b \| g(t) \|^2_{L_2(\Omega)} dt\right\}^{\frac{1}{2}}\left\{\int_a^b \| f(t) \|^2_{L_2(\Omega)} dt\right\}^{\frac{1}{2}} + \| f(a) \|^2_{L_2(\Omega)}.$$

Thus from the above inequalities and Theorem 1, by a limit process analogous as one used in my former paper [6] we see the following

Theorem 2. *For any* $g \in \mathfrak{L}^1(L_2(\Omega))$ *such that* $g(t)=0 \quad t<a$ *and for any* $b>a$, *there is a unique solution* $f(t) \in \mathfrak{L}^1(L_2(\Omega))[a, b] \frown \mathfrak{L}^0(V)[a, b] \frown C_t(L_2(\Omega))[a, b]$ *such that*

$$f(t) \in D(\bar{A}_t) \quad (a.e.t)$$

$$\left(\frac{d}{dt}+\bar{A}_t\right)f(t)=g(t) \quad (a.e.t) \ in \ L_2(\Omega)$$

$$f(a)=0 \quad in \quad L_2(\Omega).$$

Here we remark that if $(((u, v)))_t$ satisfy furthermore a condition with respect to perturbations (see § 3), then Theorem 2· can be strengthened.

3. Example (Fokker-Planck's equations). For the sake of simplicity let Ω be a bounded domain with sufficiently smooth boundary S in the Euclidean space $R^N (N \geqq 2)$. Let A_t be the following differential operator: for any sufficiently smooth function f,

$$A_t f = \frac{\partial^2}{\partial x^i \partial x^j}(b^{ij}(t, x)f(x)) + \frac{\partial}{\partial x^i}(-a^i(t, x)f(x)) \text{ on } \Omega$$

$$f \in D(A_t) \text{ if and only if}$$

$$b^{ij}(t, x)\frac{\partial f}{\partial x^j}\pi^i(x) + \left(\frac{\partial b^{ij}}{\partial x^j}(t, x) - a^i(t, x)\right)\pi^i(x)f(x)(=B(f, 1))=0 \text{ on } S,$$

where $b^{ij}(t, x)$, $a^i(t, x)$ are sufficiently smooth real functions defined on $[0, T] \times \bar{\Omega}$, $\pi^i(x)=\cos(n(x), x^i)$ on S.

Furthermore we assume that

$$b^{ij}(t, x)\xi_i \xi_j > 0$$

for any real $\xi_i : \sqrt{\Sigma \xi_i^2} \neq 0$ and for $(t, x) \in [0, T] \times \bar{\Omega}$. Then we see that for any $f \in D(A_t) \frown C^2(\bar{\Omega})$, $v \in C^1(\bar{\Omega})$,

$$\begin{aligned}(A_t f, v)_{L_2(\Omega)} = &-\left(b^{ij}(t, x)\frac{\partial}{\partial x^j}f(x), \frac{\partial}{\partial x^i}v(x)\right)_{L_2(\Omega)} \\ &+\left(a^i(t, x)f(x), \frac{\partial}{\partial x^i}v(x)\right)_{L_2(\Omega)} \\ &-\left(\frac{\partial b^{ij}}{\partial x^j}(t, x)f, \frac{\partial}{\partial x^i}v(x)\right)_{L_2(\Omega)} \\ &(+(B(f, 1), v)_{L_2(S)}).\end{aligned}$$

Let $(((u, v)))_t$ be the following:

$$\left(b^{ij}(t, x)\frac{\partial}{\partial x^j}u(x), \frac{\partial}{\partial x^i}v(x)\right)_{L_2(\Omega)} + \left(u(x), \left(-a^i(t, x)+\frac{\partial b^{ij}}{\partial x^j}(t, x)\right)\cdot\frac{\partial}{\partial x^i}v\right)_{L_2(\Omega)}.$$

Then, using some extending b^{ij}, a^i, we see that the bilinear form

$(((u, v)))_t$ satisfies the condition in §2, in fact, that in the end of §2, with $V=H^1(\bar{\Omega})$.

Furthermore we see the following

Lemma 4. *Let $u(t)\in\mathfrak{L}^1(L_2(\Omega))$ be a solution such that*

$$\left(\frac{\partial}{\partial t}-A_t\right)u(t)=0 \quad (a.e.t)$$

$$u(t)\in D(\bar{A}_t) \quad (a.e.t),$$

then $u(s)\geqq 0$ whenever $u(t)\geqq 0$ for some $t(s>t)$.

For, let $h(t, x)$ be the following function such that $h(t, x)=1, -1$ and 0 when $u(t, x)>0$, <0 and $=0$ respectively.

Then for $b>a$

$$\int_a^b \left\|\left(\frac{\partial}{\partial t}+A\right)u\right\|_{L_1(\Omega)} dt \geqq \int_a^b\int_\Omega h(t, x)\cdot\frac{\partial}{\partial t}u(t, x)\,dx\,dt$$
$$-\int_a^b\int_\Omega h(t, x)\cdot A(t, x)u(t, x)\,dx\,dt.$$

Furthermore by the regularity of solutions of elliptic equations on the boundary [1, 5] and by Yosida's lemma [8] we see that

$$\int_\Omega h(t, x)\cdot A(t, x)f(t, x)\,dx \leqq 0 \quad \text{(a.e.t).}$$

Therefore $0\geqq\int_\Omega\int_a^b h(t, x)\cdot\frac{\partial}{\partial t}u(t, x)\,dt\,dx=\|f(b)\|_{L_1(\Omega)}-\|f(a)\|_{L_1(\Omega)}$.

Furthermore from the type of A we see that

$$\int_\Omega f(b)\,dx=\int_\Omega f(a)\,dx.$$

Thus we see that the mapping $u(t)\to u(s)$ $(s>t)$ is norm preserving with respect to $L_1(\Omega)$ and $u(s)\geqq 0$, whenever $u(t)\geqq 0$.

Now let $T_{s,t}(u)=u(s)$ $(s>t)$ when $u(s)$ is a solution of our equation such that $u(t)=u$ and $u(s)\in\mathfrak{L}^1(L_2(\Omega))(t, T)$. Then from Lemma 4 and Theorem 2, we see that these $T_{s,t}$ are extended over $L_1(\Omega)$ into itself and that these extensions are transition operators on $L_1(\Omega)$. Therefore by the hypoellipticity of parabolic equations [6] and the regularity of the solutions of elliptic equations on the boundary [1, 5] we see the following

Theorem 3. *The diffusion problem with the differential operators A_t as generating operators can be solved, i.e. there is a transition density $p(t, s, x, y)$ such that for $s>t$ and for $f\in L_1(\Omega)$ setting*

$$T_{s,t}f(y)=\int_\Omega p(t, s, x, y)f(x)\,dx,$$

$T_{s,t}$ is a transition operator $[0<t<s<T)$ and such that for any $\tilde{f}\in D(\bar{A}_t)$

$$\left(\frac{\partial}{\partial t}-A_s\right)T_{s,t}\tilde{f}=0 \quad \textit{on} \quad [0, T)\times\Omega$$

$$T_{t,t}\tilde{f}=\tilde{f} \qquad on \quad L_2(\Omega)$$
$$B_s(T_{s,t}\tilde{f}, 1)=0 \qquad (a.e.\ x \in S)\ for\ all\ s \in [0, T).$$

Finally we remark that our consideration can be applicable to other diffusion problems.

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