110. On Complete Orthonormal Sets in Hilbert Space

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It is well known that a set in a separable Hilbert space*) is complete, if the set is sufficiently near a complete orthonormal set under some additional conditions. Such theorems were obtained by Paley, Wiener [7], Bellman [3] and Pollard [8] in United States, and Bary [1, 2], Kostyučenko and Skorohod [6] in Soviet Russia, and Hilding [4, 5] in Sweden.

Kostyučenko and Skorohod have given a simple proof of Bary theorem: if $\{\varphi_n\}$ and $\{\psi_n\}$ are orthonormal systems in Hilbert space, and if $\sum_{n=1}^{\infty}\|\varphi_n-\psi_n\|<\infty$, then both systems are complete, if one is. S. H. Hilding [4] has shown that, *if* $\{\varphi_n\}$ *is a complete orthonormal system and if* $\sum_{n=1}^{\infty}\|\varphi_n-\psi_n\|<1$, *then* $\{\psi_n\}$ *is complete,* and he has also obtained other two results; *let* $\{\varphi_n\}$ *be a complete orthonormal system, and let* $r_n=\|\varphi_n-\psi_n\|$,

1) *if* $\|\psi_n\|=1$ *for* $n=1,2,\cdots$ *and if* $\sum_{n=1}^{\infty} r_n^2\left(1-\frac{r_n^2}{4}\right)<1$,

or

2) *if* $(\varphi_n, \psi_n)=0$ *and if* $\sum_{n=1}^{\infty}\frac{r_n^2}{1+r_n^2}<1$,

then $\{\psi_n\}$ *is complete.*

We shall prove the following

Theorem. Let $\{\varphi_n\}$ *and* $\{\psi_n\}$ *be two orthonormal systems, let* $r_n=\|\varphi_n-\psi_n\|$.

1) *If* $\{\varphi_n\}$ *is complete,* $\|\psi_n\|=1$ *for* $n=1,2,\cdots$ *and* $\sum_{n=1}^{\infty} r_n\left(1-\frac{r_n^2}{4}\right)<\infty$, *then* $\{\psi_n\}$ *is complete.*

2) *if* $\{\varphi_n\}$ *is complete,* $(\varphi_n, \varphi_n-\psi_n)=0$ *for* $n=1,2,\cdots$, *and* $\sum_{n=1}^{\infty}\frac{r_n^2}{1+r_n^2}<\infty$, *then* $\{\psi_n\}$ *is complete.*

To prove Theorem, we shall use the techniques by S. H. Hilding [4], Kostyučenko and Skorohod [6]. First, we shall prove the second part of Theorem. Since the series $\sum_{n=1}^{\infty}\frac{r_n^2}{1+r_n^2}$ converges, there is an integer N such that

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*) For fundamental concepts, see B. Sz-Nagy: Spektraldarstellung linearer Transformationen des Hilbertschen Raumes, Berlin (1942).

$$\sum_{n=N+1}^{\infty} \frac{r_n^2}{1+r_n^2} < 1.$$

Let $\mathfrak{M}_N$ and $\mathfrak{M}'_N$ be the closed linear spaces generated by $\varphi_N, \varphi_{N+1}, \cdots$ and $\psi_N, \psi_{N+1}, \cdots$ respectively, and let $\mathfrak{N}_N$ and $\mathfrak{N}'_N$ be the orthocomplements of $\mathfrak{M}_N$ and $\mathfrak{M}'_N$. The projection $P(\mathfrak{M}'_N)$ of $\mathfrak{M}'_N$ in $\mathfrak{M}_N$ coincides with $\mathfrak{M}_N$. To prove it, suppose $P(\mathfrak{M}'_N) \neq \mathfrak{M}_N$, then there is a non-zero element g such that $\mathfrak{M}_N \ni g$, $P(\mathfrak{M}'_N) \perp g$ and $\|g\|=1$. Since P is a projection, $g \perp \mathfrak{M}_N$. Now, by $\mathfrak{M}_N \ni g$, $g = \sum_{n=N+1}^{\infty} (g, \varphi_n)\varphi_n$ and, for some $n \geq N+1$, by $(\varphi_n, \varphi_n - \psi_n) = 0$

$$\begin{aligned}
|(g, \varphi_n)|^2 &= |(g, \varphi_n - \psi_n)|^2 = \Big|\sum_{k=N+1}^{\infty} (g, \varphi_k)(\varphi_k, \varphi_n - \psi_n)\Big|^2 \\
&= \Big|\sum_{\substack{k=N+1 \\ k \neq n}}^{\infty} (g, \varphi_k)(\varphi_k, \varphi_n - \psi_n)\Big|^2 \\
&\leq \sum_{\substack{k=N+1 \\ k \neq n}}^{\infty} |(g, \varphi_k)|^2 \sum_{\substack{k=N+1 \\ k \neq n}}^{\infty} |(\varphi_k, \varphi_n - \psi_n)|^2 \\
&\leq (\|g\|^2 - |(g, \varphi_n)|^2) r_n^2.
\end{aligned}$$

Therefore we have $|(g, \varphi_n)|^2 \leq \frac{r_n^2}{1+r_n^2}$. This shows $1 = \|g\|^2 = \sum_{n=N+1}^{\infty} |(g, \varphi_n)|^2 \leq \sum_{n=N+1}^{\infty} \frac{r_n^2}{1+r_n^2}$ which contradicts $\sum_{n=N+1}^{\infty} \frac{r_n^2}{1+r_n^2} < 1$. Hence g is the zero-element, and we have $P(\mathfrak{M}'_N) = \mathfrak{M}_N$.

It is sufficient to prove that the projection $P(\mathfrak{N}_N)$ of $\mathfrak{N}_N$ in $\mathfrak{N}'_N$ coincides with $\mathfrak{N}'_N$, since $\{\psi_1, \cdots, \psi_N\}$ spans N-dimensional space by its orthonormality.

Suppose that $P(\mathfrak{N}_N) \neq \mathfrak{N}'_N$, then there is a non-zero element g such that $g \in \mathfrak{N}'_N$ and $g \perp P(\mathfrak{N}_N)$. Therefore $g \perp \mathfrak{N}_N$, and we have $g \in \mathfrak{M}_N$. On the other hand $g \perp \mathfrak{M}'_N$. By the result of the above paragraph, the element g must be zero. Hence $P(\mathfrak{N}_N) = \mathfrak{N}'_N$, and $\{\psi_1, \cdots, \psi_N\}$ spans $\mathfrak{N}'_N$. Therefore $\{\psi_n\}$ is complete.

For the first part, the quite similar method is available. Since the series $\sum_{n=1}^{\infty} r_n^2\left(1-\frac{r_n^2}{4}\right)$ is convergent, we can find an integer N such that $\sum_{n=N+1}^{\infty} r_n^2\left(1-\frac{r_n^2}{4}\right) < 1$. Using the same symbols, we can prove that the projection of $\mathfrak{M}'_N$ on $\mathfrak{M}_N$ is $\mathfrak{M}_N$.

If $P(\mathfrak{M}'_N) \neq \mathfrak{M}_N$, there is a non-zero element g such that $g \in \mathfrak{M}_N$, $g \perp P(\mathfrak{M}'_N)$ and $\|g\|=1$. Then $g \perp \mathfrak{M}'_N$ and by the hypothesis, we can prove

$$|(g, \varphi_n)| \leq r_n^2\left(1-\frac{r_n^2}{4}\right)$$

for $N+1 \leq n$ (for the detail calculation, see S. H. Hilding [4, p. 2; 5, p. 29]). Therefore $1 = \|g\|^2 \leq \sum_{n=N+1}^{\infty} |(g, \varphi_n)|^2 \leq \sum_{n=N+1}^{\infty} r_n^2\left(1-\frac{r_n^2}{4}\right) < 1$

implies $g=0$. By the technique of the first part, we can prove the completeness of $\{\psi_n\}$. Therefore we have a proof of Theorem.

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