109. On Imbedding a Metric Space in a Product of One-dimensional Spaces

By Jun-iti NAGATA
Department of Mathematics, Osaka City University
(Comm. by K. KUNUGI, M.J.A., Oct. 12, 1957)

It is well known that every separable metric space can be imbedded in Hilbert cube I^{ω}. Recently K. Morita has proved that a regular space having σ-star-finite basis can be imbedded in the topological product $N(\Omega)\times I^{\omega}$ of a generalized Baire's zero-dimensional space $N(\Omega)$ and I^{ω}.[1] On the other hand the author has shown that every n-dimensional metric space can be imbedded in a product of n+1 one-dimensional spaces.[2] However, it seems that there is little study on imbedding general metric spaces in a product of one-dimensional spaces. The purpose of this note is to show that every metric space can be imbedded in a product of countably many one-dimensional spaces.

In this note we concern ourselves only with metric spaces and mean by a covering an "open" covering.

Lemma 1. *For every covering* $\mathfrak{U}$ *of a metric space* R *there exist collections* $\mathfrak{U}_i$ $(i=1,2,\cdots)$ *of open sets and a covering* $\mathfrak{V}$ *such that* $\mathfrak{V}<\bigcup_{i=1}^{\infty}\mathfrak{U}_i<\mathfrak{U}$ *and such that each* $S^2(p,\mathfrak{V})$ $(p\in R)$ *intersects at most one set of* $\mathfrak{U}_i$ *for a fixed* i *and finitely many sets of* $\bigcup_{i=1}^{\infty}\mathfrak{U}_i$

Proof. As it was shown, for every fully normal space, by A. H. Stone,[3] there exist open collections $\mathfrak{U}_i$ $(i=1,2,\cdots)$ and a covering $\mathfrak{W}$ such that $\mathfrak{W}<\bigcup_{i=1}^{\infty}\mathfrak{U}_i<\mathfrak{U}$ and such that each set of $\mathfrak{W}$ intersects at most one set of $\mathfrak{U}_i$ and finitely many sets of $\bigcup_{i=1}^{\infty}\mathfrak{U}_i$. If we take a covering $\mathfrak{V}$ satisfying $\mathfrak{V}^{\triangle\triangle}<\mathfrak{W}$, then all the conditions of this lemma are satisfied.

Lemma 2. *For every coverings* $\mathfrak{P}_i$ $(i=1,2,\cdots)$ *with order* $\mathfrak{P}_i\leqq 2$ *and* $\mathfrak{V}$ *satisfying* $\mathfrak{V}<\bigwedge_{i=1}^{\infty}\mathfrak{P}_i$, *there exist locally finite coverings* $\mathfrak{N}_i$ $(i=1,2,\cdots)$ *such that* $\mathfrak{N}_i^*<\mathfrak{P}_i$, *order* $N_i\leqq 2$ $(i=1,2,\cdots)$ *and such that there exists a covering* $\mathfrak{W}$ *satisfying* $\mathfrak{W}<\bigwedge_{i=1}^{\infty}\mathfrak{N}_i$.

---

1) The proof of this theorem is unpublished. Cf. K. Morita: Normal families and dimension theory for metric spaces, Math. Ann., **128** (1954). Cf. also J. Nagata: On imbedding theorem for non-separable metric spaces, Jour. Inst. Polytech. Osaka City Univ., **8**, no. 1 (1957).

2) Note on dimension theory, Proc. Japan Acad., **32**, no. 8 (1956).

3) A. H. Stone: Paracompactness and product spaces, Bull. Amer. Math. Soc., **54**, no. 10 (1948).

Proof. Let $\mathfrak{U}^* < \mathfrak{V} < \bigwedge_{i=1}^{\infty} \mathfrak{P}_i$, $\mathfrak{P}_i = \{P_{\delta,i} \mid \delta \in D_i\}$, then we define coverings $\mathfrak{M}_i$ $(i=1,2,\cdots)$ by

$$\mathfrak{M}_i = \{M_{\delta,i} \mid \delta \in D_i\},\ M_{\delta,i} \smile \{U \mid S(U,\mathfrak{U}) \subseteq P_{\delta,i},\ U \in \mathfrak{U}\}.$$

First, we notice that $\mathfrak{U}^* < \bigwedge_{i=1}^{\infty} \mathfrak{P}_i$ implies $\mathfrak{U} < \bigwedge_{i=1}^{\infty} \mathfrak{M}_i$. Since each set of $\mathfrak{U}$ intersects, from order $\mathfrak{P}_i \leqq 2$, at most two sets of $\mathfrak{M}_i$, $\mathfrak{M}_i$ is a locally finite covering with order $\mathfrak{M}_i \leqq 2$.

Taking $\mathfrak{U}'$ satisfying $(\mathfrak{U}')^{\triangle} < \mathfrak{U}$, we define coverings $\mathfrak{Q}_i$ $(i=1,2,\cdots)$ by

$$L_{\delta,i} = S(M_{\delta,i} - \smile \{M_{\delta',i} \mid \delta \neq \delta' \in D_i\}, \mathfrak{U}')$$
$$Q_{\delta,i} = L_{\delta,i} - \smile \{\bar{L}_{\delta',i} \mid \delta \neq \delta' \in D_i\} \quad (\delta \in D_i),$$
$$\mathfrak{Q}_i = \{Q_{\delta,i},\ M_{\alpha,i} \frown M_{\beta,i} \mid \delta, \alpha, \beta \in D_i\}.$$

To prove $\mathfrak{U}' < \bigwedge_{i=1}^{\infty} \mathfrak{Q}_i$ we consider an arbitrary $U' \in \mathfrak{U}'$ and $\mathfrak{Q}_i$. Let $U' \subseteq M_{\delta,i} \in \mathfrak{M}_i$. In the case that $U' \not\subseteq M_{\delta',i}$ for every δ' with $\delta \neq \delta' \in D_i$, we have $U' \not\subseteq \smile \{M_{\delta',i} \mid \delta \neq \delta' \in D_i\}$. For if $U' \subseteq \smile \{M_{\delta',i} \mid \delta \neq \delta' \in D_i\}$, then U' intersects at least two of $M_{\delta',i}$ $(\delta' \neq \delta)$, which contradicts the fact that every set of $\mathfrak{U}$ intersects at most two sets of $\mathfrak{M}_i$. Therefore $U' \subseteq L_{\delta,i}$. To show $U' \frown \bar{L}_{\delta',i} = \phi$ for every $\delta' \neq \delta$, we assume the contrary, i.e. $U' \frown \bar{L}_{\delta',i} \neq \phi$, $\delta' \neq \delta$. Then there exists $U'' \in \mathfrak{U}'$ such that $U' \frown U'' \neq \phi$, $U'' \not\subseteq M_{\delta,i}$. Hence it follows from $U' \not\subseteq M_{\delta',i} (\delta' \neq \delta)$ that $U' \smile U'' \not\subseteq M_{\delta,i}$ for every $\delta \in D_i$, which contradicts $(\mathfrak{U}')^{\triangle} < \mathfrak{M}_i$. Thus we have $U' \frown \bar{L}_{\delta',i} = \phi$ $(\delta' \neq \delta)$ and consequently $U' \subseteq Q_{\delta,i} \in \mathfrak{Q}_i$.

In the case that $U' \subseteq M_{\delta',i}$, $\delta' \neq \delta$, we have $U' \subseteq M_{\delta,i} \frown M_{\delta',i} \in \mathfrak{Q}_i$. In consequence we conclude $\mathfrak{U}' < \bigwedge_{i=1}^{\infty} \mathfrak{Q}_i$.

Since $Q_{\delta,i} \frown Q_{\delta',i} = \phi$ $(\delta \neq \delta')$ is obvious from the definition of $Q_{\delta,i}$, it follows from order $\mathfrak{M}_i \leqq 2$ that order $\mathfrak{Q} \leqq 2$. If $Q_{\delta,i} \frown (M_{\alpha,i} \frown M_{\beta,i}) \neq \phi$, then $S(M_{\delta,i}, \mathfrak{U}') \frown (M_{\alpha,i} \frown M_{\beta,i}) \neq \phi$, and hence $\delta = \alpha$ or $\delta = \beta$. For example, let $\delta = \alpha$, then $Q_{\delta,i} \smile (M_{\alpha,i} \frown M_{\beta,i}) = Q_{\alpha,i} \smile (M_{\alpha,i} \frown M_{\beta,i}) \subseteq S(M_{\alpha,i}, \mathfrak{U}') \subseteq P_{\alpha,i}$. Since $Q_{\delta,i} \subseteq P_{\delta,i}$ and $M_{\alpha,i} \frown M_{\beta,i}$ are obvious, we have $\mathfrak{Q}_i^{\triangle} < \mathfrak{P}_i$. The local finiteness of $\mathfrak{Q}_i$ is obvious by the above discussion.

Repeating such a process we get locally finite coverings $\mathfrak{N}_i$ $(i=1,2,\cdots)$ such that $\mathfrak{N}_i^{\triangle} < \mathfrak{Q}_i$, order $\mathfrak{N}_i \leqq 2$ and such that there exists a covering $\mathfrak{W}$ satisfying $\mathfrak{W} < \bigwedge_{i=1}^{\infty} \mathfrak{N}_i$. Since $\mathfrak{N}_i^* < \mathfrak{P}_i$ is clear, these $\mathfrak{N}_i$ satisfy the conditions of this lemma.

Lemma 3. *Let $\mathfrak{S}_1 > \mathfrak{S}_2^* > \mathfrak{S}_2 > \mathfrak{S}_3^* > \cdots$ be a sequence of coverings of a metric space R such that $\{S(p, \mathfrak{S}_m) \mid m=1,2,\cdots\}$ is a nbd (= neighborhood) basis of each point p of R. Then there exist countably many sequences*

$$\mathfrak{N}_{1,i} > \mathfrak{N}^*_{2,i} > \mathfrak{N}_{2,i} > \mathfrak{N}^*_{3,i} > \cdots \qquad (i=1,2,\cdots)$$

of coverings such that order $\mathfrak{N}_{m,i} \leqq 2$ $(m, i=1,2,\cdots)$, *for every* m *and every point* p *of* R *there exists* $\mathfrak{N}_{m,i}$ *satisfying* $S(p, \mathfrak{N}_{m,i}) \subseteq S(p, \mathfrak{S}_m)$ *and such that for every* m *there exists a covering* $\mathfrak{W}_m$ *with* $\mathfrak{W}_m < \bigwedge_{i=1}^{\infty} \mathfrak{N}_{m,i}$.

Proof. First, we choose, for $\mathfrak{S}_2$, open collections $\mathfrak{U}_{1,i}$ and a covering $\mathfrak{V}$ satisfying $\mathfrak{S}_2 > \smile_{i=1}^{\infty} \mathfrak{U}_{1,i} > \mathfrak{V}$ and the other conditions of Lemma 1. Let $\mathfrak{U}_{1,i} = \{U_\alpha \mid \alpha \in A\}$ for a fixed i, then we define a covering $\mathfrak{N}_{1,i}$ by

$$\mathfrak{N}_{1,i} = \{S(U_\alpha, \mathfrak{V}),\ R - \smile_{\alpha \in A} \overline{U}_\alpha \mid \alpha \in A\}.$$

Let us show that $\mathfrak{N}_{1,i}$ satisfies the conditions of this lemma. Order $\mathfrak{N}_{1,i} \leqq 2$ is deduced from the fact that $S^2(p, \mathfrak{V})$ intersects at most one set of $\mathfrak{U}_{1,i}$. $\mathfrak{V} < \bigwedge_{i=1}^{\infty} \mathfrak{N}_{1,i}$ is obvious. Since $\smile_{i=1}^{\infty} \mathfrak{U}_{1,i}$ covers R, we can take, for every point p of R, i and $\alpha \in A$ such that $p \in U_\alpha \in \mathfrak{U}_{1,i}$. If $p \in S(U_\alpha, \mathfrak{V})$, then we have, from $\mathfrak{V} < \mathfrak{S}_2$, $\mathfrak{U}_{1,i} < \mathfrak{S}_2$ and $\mathfrak{S}_2^* < \mathfrak{S}_1$, $S(U_{\alpha'}, \mathfrak{V}) \subseteq S(p, \mathfrak{S}_1)$. This combining with $p \notin R - \smile_{\alpha \in A} \overline{U}_\alpha$ implies $S(p, \mathfrak{N}_{1,i}) \subseteq S(p, \mathfrak{S}_1)$.

Let us assume that we can define such $\mathfrak{N}_{l,i}$ $(i=1,2,\cdots)$ for $l \leqq m$, then we shall define $\mathfrak{N}_{m+1,i}$ $(i=1,2,\cdots)$ as follows. Since order $\mathfrak{N}_{m,i} \leqq 2$ and $\mathfrak{N} < \bigwedge_{i=1}^{\infty} \mathfrak{N}_{m,i}$ for some covering $\mathfrak{N}$, we can choose, by Lemma 2, locally finite coverings $\mathfrak{N}_i$ $(i=1,2,\cdots)$ satisfying $\mathfrak{N}_i^* < \mathfrak{N}_{m,i}$, order $\mathfrak{N}_i \leqq 2$ and $\mathfrak{N}' < \bigwedge_{i=1}^{\infty} \mathfrak{N}_i$ for some covering $\mathfrak{N}'$. Moreover there exist, by Lemma 1, open collections $\mathfrak{P}_i$ $(i=1,2,\cdots)$ and a covering $\mathfrak{Q}$ such that $\mathfrak{Q} < \bigvee_{i=1}^{\infty} \mathfrak{P}_i < \mathfrak{M}_\wedge \mathfrak{S}_{m+2}$ for a covering $\mathfrak{M}$ with $\mathfrak{M}^{**} < \mathfrak{N}' < \bigwedge_{i=1}^{\infty} \mathfrak{N}_i$ and such that each $S^2(p, \mathfrak{Q})$ intersects at most one set of $\mathfrak{P}_i$ and finitely many sets of $\smile_{i=1}^{\infty} \mathfrak{P}_i$. Let $\mathfrak{P}_i = \{P_{\beta,i} \mid \beta \in B_i\}$, $\mathfrak{N}_i = \{N_{\gamma,i} \mid \gamma < \tau_i\}$, then we denote by $\gamma = \gamma(\beta)$ the first ordinal γ such that $\overline{S(P_{\beta,i}, \mathfrak{Q})} \subseteq N_{\gamma,i} \in \mathfrak{N}_i$. Now we define coverings $\mathfrak{N}_{m+1,i}$ $(i=1,2,\cdots)$ by

$$\mathfrak{N}_{m+1,i} = \{K_{\gamma,i},\ S(P_{\beta,i}, \mathfrak{Q}) \mid \gamma < \tau_i,\ \beta \in B_i\},$$

$$K_{\gamma,i} = N_{\gamma,i} - \smile\{\overline{P}_{\beta,i} \mid \gamma = \gamma(\beta)\} \smile \{\overline{S(P_{\beta,i}, \mathfrak{Q})} \mid \gamma \neq \gamma(\beta)\}.$$

First, $\mathfrak{N}_{m+1,i} < \mathfrak{N}_i < \mathfrak{N}_i^* < \mathfrak{N}_{m,i}$ is obvious from $\mathfrak{Q} < \mathfrak{P}_i < \mathfrak{M} < \mathfrak{M}^{**} < \mathfrak{N}_i$. To show order $\mathfrak{N}_{m+1,i} \leqq 2$, we take an arbitrary point p of R. If $p \notin S(P_{\beta,i}, \mathfrak{Q})$ for every $\beta \in B_i$, then p is contained, by order $\mathfrak{N}_i \leqq 2$, in at most two of $K_{\gamma,i} (\gamma < \tau_i)$. If $p \in S(P_{\beta,i}, \mathfrak{Q})$, then it follows from the relation of $\mathfrak{Q}$ and $\mathfrak{P}_i$ that $p \notin S(P_{\alpha,i}, \mathfrak{Q})$ for every α with $\beta \neq \alpha \in B_i$. Since it follows from the definition of $K_{\gamma,i}$ that $p \notin K_{\gamma,i}$ for every $\gamma < \tau_i$ with $\gamma \neq \gamma(\beta)$, p is contained in at most two sets of $\mathfrak{N}_{m+1,i}$. Therefore we have order $\mathfrak{N}_{m+1,i} \leqq 2$.

We notice that $\mathfrak{N}_{m+1,i}$ covers R. For if $p \notin S(P_{\beta,i}, \mathfrak{Q})$ for every $\beta \in B_i$, $p \notin \overline{S(P_{\beta,i}, \mathfrak{Q})}$ $(\beta \in B_i)$ implies $p \in K_{\gamma,i}$ for γ satisfying $p \in N_{\gamma,i}$. On the other hand $p \in \overline{S(P_{\beta,i}, \mathfrak{Q})}$ implies $p \in K_{\gamma,i}$ for $\gamma = \gamma(\beta)$, proving $\mathfrak{N}_{m+1,i}$ covers R.

Next there exists a covering $\mathfrak{V}$ such that $\mathfrak{V} < \bigwedge_{i=1}^{\infty} \mathfrak{N}_{m+1,i}$. Denoting by p an arbitrary point of R, we see that $S(p, \mathfrak{Q})$ intersects a finite number of $\overline{S(P_{\beta,i}, \mathfrak{Q})}$ $(\beta \in B_i,\ i=1,2,\cdots)$. We assume $S(p, \mathfrak{Q})$ intersects $S(P_{\beta(i),i}, \mathfrak{Q})$ $(i=i_1,\cdots,i_s)$ only. Then for $i \neq i_1,\cdots,i_s$ we have, from $\mathfrak{Q}^{\triangle} < \mathfrak{N}_i$, $S(p, \mathfrak{Q}) \subseteq K_{\gamma,i}$ for some $\gamma < \tau_i$. Hence there exist an open nbd $U(p)$ of p and $N_i \in \mathfrak{N}_{m+1,i}$ $(i=1,2,\cdots)$ satisfying $U(p) \subseteq \bigcap_{i=1}^{\infty} N_i$.

Last we take, for a given point p of R, i and $\beta \in B_i$ such that $p \in P_{\beta,i}$. Then it follows from $\mathfrak{Q}, \mathfrak{P}_i < \mathfrak{S}_{m+2} < \mathfrak{S}^*_{m+2} < \mathfrak{S}_{m+1}$ that $S(p, \mathfrak{N}_{m+1,i}) = S(P_{\beta,i}, \mathfrak{Q}) \subseteq S(p, \mathfrak{S}_{m+1})$. Thus $\mathfrak{N}_{m+1,i}$ $(i=1,2,\cdots)$ satisfy all the desired conditions.

Lemma 4. *Every metric space has sequences*

$$\mathfrak{U}_{1,i} > \mathfrak{U}^{**}_{2,i} > \mathfrak{U}_{2,i} > \mathfrak{U}^{**}_{3,i} > \cdots \quad (i=1,2,\cdots)$$

of coverings such that $S(p, \mathfrak{U}_{m+1,i})$ *intersects at most two sets of* $\mathfrak{U}_{m,i}$ *and such that* $\{S(p, \mathfrak{U}_{m,i}) \mid m, i=1,2,\cdots\}$ *is a nbd basis of* p.

Proof. We can deduce this lemma directly from Lemma 3 as we have shown in the previous paper.[4]

Theorem. *Every metric space* R *can be topologically imbedded in a product of an enumerable number of functional spaces* R_i *with* $\dim R_i \leqq 1$ $(i=1,2,\cdots)$.

Proof. The proof of this theorem is analogous to the previous one.[5] Let us sketch the outline of the proof. We denote by $\mathfrak{U}_{1,i} > \mathfrak{U}^{**}_{2,i} > \mathfrak{U}_{2,i} > \mathfrak{U}^{**}_{3,i} > \cdots$ $(i=1,2,\cdots)$ the sequences satisfying the conditions of Lemma 4. Let $\mathfrak{U}_{m,i} = \{U_\alpha \mid \alpha \in A_{m,i}\}$, $V_\alpha = S(U_\alpha, \mathfrak{U}_{m+1,i})$ $(\alpha \in A_{m,i})$, then we can define continuous functions $f_{\alpha,m,i}$ $(\alpha \in A_{m,i})$ such that $f_{\alpha,m,i}(V^c_\alpha) = 0$, $f_{\alpha,m,i}(U_\alpha) = 1/2^{m-1}$ $(\alpha \in A_{m,i})$, $0 \leqq f_{\alpha,m,i} \leqq 1/2^{m-1}$ and such that $y \in S(x, \mathfrak{U}_{l,i})$ implies $|f_{\alpha,m,i}(x) - f_{\alpha,m,i}(y)| < A/2^l$ for some definite number A and for every m and $\alpha \in A_{m,i}$. Considering a topological product $P_i = \Pi\{I_\alpha \mid \alpha \in A_{m,i}, m=1,2,\cdots\}$ of $I_\alpha = \{x \mid 0 \leqq x \leqq 1/2^{m-1}\}$ $(\alpha \in A_{m,i})$, we define a mapping F_i of R into P_i by

$$F_i(x) = \{f_{\alpha,m,i}(x) \mid f_{\alpha,m,i}(x) \in I_\alpha \ (\alpha \in A_{m,i}, m=1,2,\cdots)\} \ (x \in R).$$

Now we can show that $R_i = F_i(R)$ $(\subseteq P_i)$ is a metrizable space with $\dim F_i(R) \leqq 1$. Letting $N_\alpha = F_i(R) \frown \{\{p_\alpha\} \mid p_\alpha > 0\}$ $(\alpha \in A_{m,i})$, $\mathfrak{N}_{m,i} = \{N_\alpha \mid \alpha \in A_{m,i}\}$ we have a covering $\mathfrak{N}_{m,i}$ of $R_i = F_i(R)$. We can show easily order $\mathfrak{N}_{m,i} \leqq 2$, $\mathfrak{N}^*_{m+1,i} < \mathfrak{N}_{m,i}$ and that $\{S(p, \mathfrak{N}_{m,i}) \mid m=1,2,\cdots\}$ is a nbd basis of each point p of R_i. Hence we can conclude, from the

4) The proof of Theorem 2 of "Note on dimension theory" loc. cit.

5) The proof of Theorem 3 of "Note on dimension theory" loc. cit.

previous theorem,[6] the metrizability of R_i and $\dim R_i \leqq 1$. As it is well known, we can regard R_i as a functional space of functions of $\bigcup_{m=1}^{\infty} A_{m,i}$, where the strong topology of R_i is clearly identical with the weak one.

Now we define a mapping $F(x)$ of R into $\prod_{i=1}^{\infty} R_i$ by $F(x)=(F_1(x), F_2(x), \cdots)$ $(x \in R)$. Then $F(x)$ is, as easily seen, a homeomorphic mapping. Thus R is homeomorphic with the subspace $F(R)$ of the product space $\prod_{i=1}^{\infty} R_i$ of functional spaces R_i with $\dim R_i \leqq 1$.

6) Note on dimension theory, Theorem 2.