107. On Generalized Walsh Fourier Series. I

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1. We shall state some theorems on generalized Walsh Fourier series, that is, on Fourier series with respect to the system of the generalized Walsh functions.

Let $\{\alpha(n)\}$ be a sequence of integers not less than 2, and put $A(0)=1$, $A(n)=\alpha(0)\alpha(1)\cdots\alpha(n-1)$, $A(-n)=1/A(n)$. The "generalized Rademacher functions" $\phi_n(t)$ $(n=0,1,2,\cdots)$ are defined as

$$\phi_n(t)=\exp(2\pi ik/\alpha(n))$$

for t belonging to the left-semiclosed intervals

$$(kA(-n-1),\ (k+1)A(-n-1)) \qquad k=0,1,\cdots,A(n+1)-1$$

and $\phi_n(t+1)=\phi_n(t)$ for all t.

Now we can define the "generalized Walsh functions" $\psi_n(t)$ $(n=0,1,2,\cdots)$. Let

$$\psi_0(t)=1$$

and for $n\geqq 1$,

$$\psi_n(t)=\phi_{n(1)}^{a(1)}(t)\,\phi_{n(2)}^{a(2)}(t)\cdots\phi_{n(r)}^{a(r)}(t)$$

provided that n is expressed in the form

$$n=a(1)A(n(1))+a(2)A(n(2))+\cdots+a(r)A(n(r))$$

where

$$n(1)>n(2)>\cdots>n(r)\geqq 0;\ 0<a(j)<\alpha(j) \quad (j=1,2,\cdots,r).$$

The functions $\psi_n(t)$ thus defined form a complete orthonormal system over the interval (0, 1). If $\alpha(n)=2$ $(n=0,1,2,\cdots)$, the system reduces to that of Walsh, and the case $\alpha(n)=\alpha$ was studied by H. E. Chrestenson [1]. The general definition seems to have been given by J. J. Price (cf. [7]), but we have not been able to know the details.

We assume in the sequel that, unless others are stated explicitly, the sequence $\{\alpha(n)\}$ is bounded, say $\alpha(n)\leqq\alpha$ $n=0,1,2,\cdots$. Though this assumption may seem stringent, it is necessary, in order to obtain positive results, to confine the "growth" of $\alpha(n)$ under a certain restriction (see Theorem 4 below).

2. The key theorem in the $L^p(p>1)$ theory of Walsh Fourier series is the following, due to R. E. A. C. Paley [6]:

Theorem P. *Let* $f(t)\in L^p(0,1)$, $f(t)\sim\sum_{n=0}^{\infty}c_n\psi_n(t)$. *Then, putting* $f_n(t)=\sum_{\nu=2^n}^{2^{n+1}-1}c_\nu\psi_\nu(t)$ $(n=0,1,2,\cdots)$, *one has*

$$B'_p \int_0^1 |f(t)|^p\,dt \leqq \int_0^1 \Big(|c_0|^2+\sum_{n=0}^{\infty}|f_n(t)|^2\Big)^{p/2} dt \leqq B''_p \int_0^1 |f(t)|^p\,dt$$

*where the constants B'_p, B''_p depend only on p.**)

The formally conceivable analogue of Theorem P holds true in our case, but it does not act so effectively; a "finer decomposition", which we propose in the following Theorem 1, would be essential for applications.

Theorem 1. *Let $f(t)\in L^p(0,1)$ $(p>1)$, $f(t)\sim\sum_{n=0}^{\infty}c_n\psi_n(t)$ and put*

$$\delta_{n,j}(t)=\sum_{\nu=jA(n)}^{(j+1)A(n)-1}c_\nu\psi_\nu(t)\qquad \begin{pmatrix} j=1,2,\cdots,\alpha(n)-1; \\ n=0,1,2,\cdots \end{pmatrix}.$$

Then we have

$$B'_{\alpha,p}\int_0^1|f(t)|^p\,dt\leqq\int_0^1\Big(|c_0|^2+\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1}|\delta_{n,j}(t)|^2\Big)^{p/2}dt\leqq B''_{\alpha,p}\int_0^1|f(t)|^p\,dt.$$

The proof of this theorem is somewhat more complicated than that of Theorem P, but runs very closely. Considering a special case in which every $\delta_{n,j}$ consists of a single term, we deduce immediately the following proposition, which is a generalization of the well-known inequalities of A. Khintchine.

Corollary. *Let $p>0$, $f(t)=\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1}c_{n,j}\phi_n^j(t)$. Then we have*

$$B'_{\alpha,p}\int_0^1|f(t)|^p\,dt\leqq\Big(\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1}|c_{n,j}|^2\Big)^{p/2}\leqq B''_{\alpha,p}\int_0^1|f(t)|^p\,dt.$$

Theorem 1 and a standard argument (see for example [6, the proof of Theorem VI]) tell us that the following theorem is valid.

Theorem 2. *Let $f(t)\in L^p(0,1)$ $(p>1)$, $f(t)\sim\sum_{n=0}^{\infty}c_n\psi_n(t)$ and let $s_n(t)=\sum_{\nu=0}^{n-1}c_\nu\psi_\nu(t)$. Then we have*

(i) $$\int_0^1|s_n(t)|^p\,dt\leqq B_{\alpha,p}\int_0^1|f(t)|^p\,dt$$

(ii) $$\int_0^1|f(t)-s_n(t)|^p\,dt\to 0\quad \textit{as } n\to\infty.$$

In case $\alpha(n)=\alpha$ $(n=0,1,2,\cdots)$, we can generalize Theorems 1 and 2 into the following

Theorem 3. *Let $p>1$, $-1/p<\gamma<1-1/p$ and suppose*

$$\int_0^1|f(t)|^p\,t^{\gamma p}\,dt<\infty,\qquad f(t)\sim\sum_{n=0}^{\infty}c_n\psi_n(t).$$

Putting now $\delta_{n,j}(t)=\sum c_\nu\psi_\nu(t)$, where the summation is extended over $j\alpha^n\leqq\nu\leqq(j+1)\alpha^n-1$, we have

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*) We use throughout the article the letter B with subscripts to denote a constant (which need not the same in different contexts) depending only on parameters disposed explicitly.

(i) $$B'_{\alpha,\gamma,p}\int_0^1 |f(t)|^p t^{\gamma p}\,dt \leqq \int_0^1 \Big(|c_0|^2+\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha-1}|\delta_{n,j}(t)|^2\Big)^{p/2} t^{\gamma p}\,dt \leqq B''_{\alpha,\gamma,p}\int_0^1 |f(t)|^p t^{\gamma p}\,dt;$$

(ii) $$\int_0^1 |s_n(t)|^p t^{\gamma p}\,dt \leqq B_{\alpha,\gamma,p}\int_0^1 |f(t)|^p\,dt;$$

(iii) $$\int_0^1 |f(t)-s_n(t)|^p t^{\gamma p}\,dt \to 0 \qquad as\ n\to\infty.$$

This, for $\alpha=2$, was proved by I. I. Hirschman [5]. His dual results (see [5, Theorems 3.2 and 4.1]) would remain true for general α, but we shall not treat them here.

To show that our assumption $\alpha(n)\leqq\alpha$ is not superfluous, we cite a negative result:

Theorem 4. *If $\alpha(n)$ increases with a gap, that is, if there is a number $\lambda>1$ such that $\alpha(n+1)/\alpha(n)\geqq\lambda$ for $n=0,1,2,\cdots$, we can find a function $f(t)$ belonging to every Lebesgue class $L^p(0,1)$, $0<p<2$ and for which $\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1}|\delta_{n,j}(t)|^2=\infty$ for all t.*

3. The Cesàro summability of Walsh Fourier series was proved by N. J. Fine [3]. Recently S. Yano [10] sharpened this into a maximal theorem and brought to the case of generalized Walsh Fourier series. In this connection we give two theorems, the one concerning summability factors and the other convergence factors.

Theorem 5. *Let $f(t)\in L(0,1)$, $f(t)\sim\sum_{n=0}^{\infty}c_n\psi_n(t)$ and let $0<\eta<1$. Then, denoting by $N_n^{(\eta)}(t;f)$ the n-th $(C,-\eta)$ mean of the series $\sum_{n=0}^{\infty}\frac{c_n\psi_n(t)}{(n+1)^{\eta}}$, we have*

(i) $$\int_0^1 \sup_n |N_n^{(\eta)}(t;f)|\,dt \leqq B_{\alpha,\eta}\int_0^1 |f(t)|\,dt;$$

(ii) *the series $\sum_{n=0}^{\infty}\frac{c_n\psi_n(t)}{(n+1)^{\eta}}$ is summable $(C,-\eta)$ almost everywhere.*

In the case of $\alpha(n)=2$ $(n=0,1,2,\cdots)$, this theorem was proved by S. Yano [9].

Theorem 6. *Let $f(t)\in L^2(0,1)$, $f(t)\sim\sum_{n=0}^{\infty}c_n\psi_n(t)$. Then, denoting by $s_n^*(t)$ the n-th partial sum of the series $\sum_{n=0}^{\infty}\frac{c_n\psi_n(t)}{\sqrt{\log(n+2)}}$, we have*

(i) $$\int_0^1 \sup_n |s_n^*(t)|^2\,dt \leqq B_\alpha\int_0^1 |f(t)|^2\,dt;$$

(ii) *the series $\sum_{n=0}^{\infty}\frac{c_n\psi_n(t)}{\sqrt{\log(n+2)}}$ converges almost everywhere.*

In the case of $\alpha(n)=2$ $(n=0, 1, 2, \cdots)$, this theorem was stated by R. E. A. C. Paley [6] and proved by S. Yano [9]. Our proof of Theorem 6 is based on a method established by G. H. Hardy and J. E. Littlewood [4], and requires a modification of a result proved by G. Sunouchi [8] under a more general form. We refer here to another treatise [2] of N. J. Fine, where is originally given the notion of the "dyadic group", which, after suitable modifications, is indispensable to our proof of Theorem 6.

Added in Proof: Theorem 4 can be ameliorated; firstly, we have only to suppose the unboundedness of the sequence $\{\alpha(n)\}$; secondly, an "opposite" proposition holds for $p>2$. More precisely, *for* $\{\alpha(n)\}$ *unbounded, we can find a function* $g(t)$, *belonging to none of the Lebesgue classes* L^p, $p>2$, *and for which* $\sum_{n=0}^{\infty}\sum_{j=1}^{\alpha(n)-1}|\delta_{n,j}(t)|^2 \leqq M$ *for all* t.

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