104. *On Homomorphisms of the Ring of Continuous Functions onto the Real Numbers*

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Let X be a C^∞-manifold and $F(X)$ be the ring of all the C^∞-functions on X, or let X be a Q-space[1] and $C(X)$ be the ring of all the real-valued continuous functions on X. Then for a non-trivial homomorphism ϕ (i.e. $\phi(f)\neq 0$) of the function ring $F(X)$ or $C(X)$ into the real number field R, there exists one and only one point p of X such that $\phi(f)=f(p)$ for any f of the respective function ring. Hence it follows that C^∞-manifolds X and Y are differentiably homeomorphic if $F(X)$ and $F(Y)$ are isomorphic,[2] and that Q-spaces X and Y are homeomorphic if $C(X)$ and $C(Y)$ are isomorphic.[3] In this paper we shall study the generalizations of these results. For brevity we use the word 'homomorphism' in place of the word 'non-trivial homomorphism'.

Let X be a completely regular space and let $C(X, R)$ be the ring of all the real-valued continuous functions on X. We denote by $\mathfrak{C}$ a subring of $C(X, R)$ satisfying the following conditions:

(1) $R\subset\mathfrak{C}$,

(2) for a closed set F of X and a point $p\notin F$, there exists a function f of $\mathfrak{C}$ such that $f(p)>\sup_{x\in F} f(x)$,

(3) if $f(x)>a>0$ and $f(x)\in\mathfrak{C}$, then $f^{-1}(x)\in\mathfrak{C}$.

The conditions (2) and (3) are weaker than the following conditions (2′) and (3′) respectively:

(2′) for a closed set F of X and a point $p\notin F$, there exists a function f of $\mathfrak{C}$ such that $0\leqq f(x)\leqq 1$, $f(p)=1$, and $f(x)=0$ if $x\in F$,

(3′) if $f(x)>0$ and $f(x)\in\mathfrak{C}$, then $f^{-1}(x)\in\mathfrak{C}$.

It is obvious that the conditions (1), (2′) and (3′) are all fulfilled, if $\mathfrak{C}=F(X)$ or $C(X)$.

We now define a uniform structure gX of X by the following uniform neighborhoods:

$U_{f_1,\cdots,f_n;\varepsilon}(x)=\{y \mid |f_i(y)-f_i(x)|<\varepsilon,\ i=1,2,\cdots,n\}$, where $f_i\in\mathfrak{C}$ $(i=1,2,\cdots,n)$ and ε is an arbitrary positive number. Then it is easily seen that gX agrees with the topology of X by virtue of (2).

1) By a C^∞-manifold we mean a separable C^∞-manifold. For the definition of a Q-space see [3, 4, 7].

2) See [1].

3) See [3, Theorem 57].

Theorem 1. *For any homomorphism ϕ of the ring $\mathfrak{C}$ into R there exists one and only one point p of X such that $\phi(f)=f(p)$ for any f of $\mathfrak{C}$, if and only if gX is complete.*

Proof. If we put $\mathfrak{R}=\{f\mid\phi(f)=0\}$, then $\mathfrak{R}$ is a maximal ideal of $\mathfrak{C}$. In fact, if $\phi(g)=a\neq 0$, $g\in\mathfrak{C}$, then we have $1-g(x)/a\in\mathfrak{R}$, since $\phi(c)=c$ for any real number c and $\phi(g/a)=\phi(g)/a=1$.[4] Hence we have $f+g/a=1$ for some $f\in\mathfrak{R}$. This shows that an ideal generated by $\mathfrak{R}$ and g contains 1, that is, $\mathfrak{R}$ is a maximal ideal. Now let $F(\mathfrak{R})=\{F_{1/n}(f)\mid f\in\mathfrak{R},\ n=1,2,\cdots\}$, where $F_{1/n}(f)=\{x\mid\ |f(x)|\leqq 1/n\}$. Then it is trivial that $F(\mathfrak{R})$ does not contain the void set by virtue of (3). Further $F(\mathfrak{R})$ has the finite intersection property. In fact, if $F_{1/n_i}(f_i)\in F(\mathfrak{R})$ $(i=1,2,\cdots,n)$ and $\bigcap_{i=1}^{n}F_{1/n_i}(f_i)=\phi$, then we have $f=\sum_{i=1}^{n}f_i^2>\min_i(1/n_i)^2>0$ and $f\in\mathfrak{R}$. Hence $\mathfrak{R}$ contains $1=ff^{-1}$, since $f^{-1}\in\mathfrak{C}$ by (3). This contradicts the maximality of the ideal $\mathfrak{R}$. Now let ε be an arbitrary positive number and let k be a positive integer such that $2/k<\varepsilon$. Then for any point x of $\bigcap_{i=1}^{n}F_{1/k}(f_i)$, we have $\bigcap_{i=1}^{n}F_{1/k}(f_i)\subset U_{f_1,\cdots,f_n;\varepsilon}(x)$, where $f_i\in\mathfrak{R}$. On the other hand, for any uniform neighborhood $U_{f_1,\cdots,f_n;\varepsilon}$ we can assume that $f_i\in\mathfrak{R}$ for every i without losing the generality, since $U_{f_1,\cdots,f_n;\varepsilon}=U_{f_1-a_1,\cdots,f_n-a_n;\varepsilon}$, where $a_i=\phi(f_i)$. Thus all the finite intersections of sets of $F(\mathfrak{R})$ form a Cauchy filter base, which converges to a point $p\in X$ by the hypothesis. Then we have $p\in Z(f)=\{x\mid f(x)=0\}$ for any $f\in\mathfrak{R}$, and $\phi(f)=f(p)$ for any $f\in\mathfrak{C}$. It is obvious that p is the unique point such that $\phi(f)=f(p)$ for any $f\in\mathfrak{C}$. To prove the converse, let $\overline{gX}$ be the completion of gX. Since every $f\in\mathfrak{C}$ is uniformly continuous on gX, every $f\in\mathfrak{C}$ is extended uniformly continuously over $\overline{gX}$. Now assume that $\overline{gX}\neq gX$. Then for any $x\in\overline{gX}-gX$, there is a homomorphism ϕ_x of $\mathfrak{C}$ into R such that $\phi_x(f)=\tilde{f}(x)$, where $\tilde{f}$ is an extension of f over $\overline{gX}$. Then there is no point y of X such that $\phi_x(f)=f(y)$ for any $f\in\mathfrak{C}$. In fact, for any y of X, there exists a uniform neighborhood U^* of $\overline{gX}$ such that $U^*(x)\frown U^*(y)=\phi$, since $\overline{gX}$ is separated. Moreover, if we put $U(y)=U^*(y)\frown X$, there exists a function f of $\mathfrak{C}$ such that $\sup_{z\notin U(y)}f(z)<f(y)$. From this it follows that $\tilde{f}(x)\leqq\sup_{z\notin U(y)}f(z)<f(y)$. Thus we complete the proof.

By the mapping $\alpha:\alpha(x)=\{f(x)\mid f\in\mathfrak{C}\}$, X is homeomorphically mapped into the Cartesian product space $\prod_{f\in\mathfrak{C}}R_f$ by virture of (2), where R_f is the space R of the real numbers for any $f\in\mathfrak{C}$. Let Y be the

4) Every homomorphism ϕ of the ring R of the real numbers into R is the identity mapping.

image of X by the mapping α. If gX is complete, then Y is a closed subspace of $\prod_{f\in\mathfrak{C}} R_f$. Hence X is the Q-space by [7, Theorem 1].

Now let X and Y be completely regular spaces, and let $\mathfrak{C}_X$ and $\mathfrak{C}_Y$ be the subrings of $C(X, R)$ and $C(Y, R)$ respectively, each of which satisfies the conditions (1), (2) and (3). Then we have the following

Theorem 2. *If the uniform spaces gX and gY, which are determined by $\mathfrak{C}_X$ and $\mathfrak{C}_Y$, are complete, and the rings $\mathfrak{C}_X$ and $\mathfrak{C}_Y$ are isomorphic, then X and Y are homeomorphic. Moreover, if η is the homeomorphic mapping from X to Y, then we have $f\eta^{-1}\in\mathfrak{C}_Y$ for any $f\in\mathfrak{C}_X$ and $f'\eta\in\mathfrak{C}_X$ for any $f'\in\mathfrak{C}_Y$, where $f\eta^{-1}(p')=f(\eta^{-1}p')$ and $f'\eta(p)=f'(\eta p)$.*

Proof. For any point p of X, let ϕ_p be a ring-homomorphism of $\mathfrak{C}_X$ into R such that $\phi_p(f)=f(p)$ for any $f\in\mathfrak{C}_X$. If we denote by ψ the isomorphic mapping from $\mathfrak{C}_Y$ onto $\mathfrak{C}_X$, then $\phi_p\psi$ is a ring-homomorphism of $\mathfrak{C}_Y$ into R. Therefore by Theorem 1, there exists one and only one point p' of Y such that $\phi_p\psi(f')=f'(p')$ for any $f'\in\mathfrak{C}_Y$. If we denote by η the mapping: $p\to p'$, then η is obviously a 1-1 mapping from X to Y, and we have $\psi(f')=f'\eta$ for any $f'\in\mathfrak{C}_Y$ and $\psi^{-1}(f)=f\eta^{-1}$ for any $f\in\mathfrak{C}_X$. Let $U(p')$ be any neighborhood of $p'\in Y$, and let f' be a function of $\mathfrak{C}_Y$ such that $f'(p')>\sup_{x\notin U(p')} f'(x)$. Then we have $V(p')=\{q' \mid f'(q')>f'(p')-\varepsilon\}\subset U(p')$ for some positive number ε, and $\eta^{-1}V(p')$ is open in X, since $\eta^{-1}V(p')=\{q \mid \psi f'(q)>\psi f'(p)-\varepsilon\}$. Thus η is a continuous mapping. η^{-1} is also a continuous mapping from Y to X. This completes the proof.

Corollary. *Let X and Y be the Q-spaces. If the rings $C(X, R)$ and $C(Y, R)$ are isomorphic, then X and Y are homeomorphic.*

Proof. The uniform spaces gX and gY, which are determined by $C(X, R)$ and $C(Y, R)$ respectively, are both complete. Hence by Theorem 2, X and Y are homeomorphic.

We shall state a sufficient condition, under which gX is complete.

Theorem 3. *Let X be a locally compact Hausdorff space such that $X=\bigcup_{n=1}^{\infty} B_n$, where each of B_n is compact, and let $\mathfrak{C}$ be a subring of $C(X, R)$ which satisfies the following condition (4) besides (1) and (2′):*

(4) *for a sequence of non-negative functions $\{f_n\}$ of $\mathfrak{C}$ such that $\{P(f_n)\}$ is locally finite, we have $\sum_{n=1}^{\infty} f_n(x)\in\mathfrak{C}$, where $P(f_n)=\{x \mid f_n(x)>0\}$.*

Then the uniform space gX determined by $\mathfrak{C}$ is complete.

Proof. Let $\mathfrak{A}=\{A_\lambda \mid \lambda\in\Lambda\}$ be the Cauchy filter in gX. We show that some A_λ of $\mathfrak{A}$ is contained in a certain compact set. Now let $U(p)$ be a neighborhood of a point p of X such that $\overline{U(p)}$ is compact.

Then the open covering $\{U(p) \mid p \in X\}$ has a locally finite open refinement $\{V_i \mid i=1, 2, \cdots\}$, since X is paracompact.[5] Furthermore it can be easily seen by the mathematical induction that there exists an open refinement $\{W_i \mid i=1, 2, \cdots\}$ of $\{V_i\}$ such that $\overline{W}_i \subset V_i$ by virtue of normality of X. Then $\{W_i\}$ is also locally finite. We note that for any positive integer i, there exists a non-negative function $f_i \in \mathfrak{C}$ such that $f_i(x)=0$ if $x \notin V_i$ and $a_i = \min_{x \in \overline{W}_i} f_i(x) > 0$. Now let $f(x) = \sum_{i=1}^{\infty} (i/a_i) f_i(x)$. Then we have $f(x) \in \mathfrak{C}$ by virtue of (4). On the other hand, for any positive number ε, there exists an $A_\lambda \in \mathfrak{A}$ such that $A_\lambda \subset U_{f;\varepsilon}(x)$ for any $x \in A_\lambda$, since $\mathfrak{A}$ is the Cauchy filter in gX. This means that $f(x)$ is bounded on A_λ. Thus A_λ must be contained in $\bigcup_{j=1}^{n} \overline{W}_j$ for some positive integer n, since $f(x) \geqq m$ if $x \in \bigcup_{j \geqq m} \overline{W}_j$. From this it follows that the Cauchy filter $\mathfrak{A}$ converges to a point of X.

Corollary 1. *If X is a C^∞-manifold and $F(X)$ is the ring of all the C^∞-functions on X, then for any homomorphism ϕ of $F(X)$ into R, there exists one and only one point p of X such that $\phi(f)=f(p)$ for any $f \in F(X)$.*

Proof. Since $F(X)$ satisfies the conditions (1), (2′) and (4) in Theorem 3, the uniform space gX determined by $F(X)$ is complete. Hence by Theorem 1 we have the desired result.

Corollary 2. *If X is a C^∞-manifold and $D(X)$ is the ring of all the C^∞-functions with compact carriers on X, then for any homomorphism ϕ of $D(X)$ into R, there exists one and only one point p of X such that $\phi(f)=f(p)$ for any $f \in D(X)$.*

This can be shown by using Corollary 1 and the partition of unity.[6]

Corollary 3. *The ring $F(X)$ ($D(X)$) characterizes the C^∞-structure of the C^∞-manifold X.*[7]

For a particular homomorphism ϕ of the ring $\mathfrak{C}$ satisfying the conditions (1), (2′) and (3) into R, we have the following

Theorem 4. *There exists one and only one point p of X such that $\phi(f)=f(p)$ for any $f \in \mathfrak{C}$, if and only if ϕ is weakly continuous on $\mathfrak{C}$ with its weak topology.*

The proof is omitted, since it can be carried by the similar way as in [5, Theorem 3].

In the case when X is a locally compact (but not compact) Hausdorff space and $\mathfrak{C}_k$ is the ring of all the real-valued continuous

5) Cf. [1, p. 17, Lemma 2].

6) This idea of the proof was communicated to the author by Mr. K. Shiga. It is also possible to prove it directly.

7) Shanks's result [6] asserting that the ring $D^k(X)$ of functions of C^k-class with compact carriers on a manifold X of C^k-class characterizes the structure of the manifold X can be proved similarly.

functions with compact carriers, $\mathfrak{C}_k$ does not satisfy the conditions (1) and (3). But we have the following

Theorem 5. *For any homomorphism* ϕ *of the ring* $\mathfrak{C}_k$ *into* R, *there exists one and only one point* $p \in X$ *such that* $\phi(f)=f(p)$ *for any* $f \in \mathfrak{C}_k$.

Proof. We note that $\phi(\alpha f)=\alpha\phi(f)$ for any $f \in \mathfrak{C}_k$ and any real number α. In fact, if we put $K=\overline{\{x \mid f(x) \neq 0\}}$, there exists a function g of $\mathfrak{C}_k$ such that $g(x)=\alpha$ if $x \in K$. Hence we have $\alpha f(x)=g(x)f(x)$ for every $x \in X$. From this it follows that $\phi(\alpha f)=0$ if $\phi(f)=0$. If $\phi(f) \neq 0$, then the mapping $\phi^*(\alpha)=\phi(\alpha f)/\phi(f)$ is the identity mapping from R onto itself, which shows that $\phi(\alpha f)=\alpha\phi(f)$. By using this fact, it can be easily seen that $\mathfrak{R}=\phi^{-1}(0)$ is a maximal ideal of $\mathfrak{C}_k$. On the other hand, an ideal $\mathfrak{J}$ in $\mathfrak{C}_k$ is maximal if and only if $\mathfrak{J}=\mathfrak{J}_p$, where $p \in X$ and $\mathfrak{J}_p=\{f \mid f(p)=0,\ f \in \mathfrak{C}_k\}$, as shown by [2, Theorem 3]. Hence we have $\mathfrak{R}=\mathfrak{J}_p$ for a point p of X. Now let $f \in \mathfrak{C}_k$ and $f(p)=1$, and assume that $\phi(f)=\lambda$. Then we have $\phi(f^2)=\lambda$, since $f(p)^2=1$. On the other hand it holds that $\phi(f^2)=\phi(f)^2=\lambda^2$. Hence we have $\lambda=1$, since $\lambda \neq 0$. For any f of $\mathfrak{C}_k$ such that $f(p)=\alpha \neq 0$, we have $\phi(f/\alpha)=1$, that is, $\phi(f)=\alpha$, since $f/\alpha(p)=1$. This completes the proof.

The following corollary is due to Shanks [6].

Corollary. *The ring* $\mathfrak{C}_k$ *characterizes the topology of the locally compact Hausdorff space* X.

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