21. Some Properties of $(n-1)$-Manifolds in n-Space

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In this note we shall give a brief account of some properties of a polyhedral $(n-1)$-manifold in the n-dimensional Euclidean space R^n, that is, of a triangulable $(n-1)$-manifold P^{n-1} rectilinearly imbedded in R^n. Theorems 1, 2, 3, 4 relate to the differentiable approximations of P^{n-1} in R^n and Theorems 5, 6 relate to the curvatura integra of P^{n-1} in R^n. Full details will appear in Osaka Mathematical Journal.

1. Let S be a point set in some Euclidean space R^n. A k-plane H^k $(k\geqq 1)$ in R^n will be called *transversal to* S if there exists a positive number ε such that a line through any two points of S makes an angle greater than ε with H^k. A k-plane $H^k(p)$ through a point p of S will be called *transversal to* S *at* p if $H^k(p)$ is transversal to some neighbourhood of p on S.

Let M^m be a topological manifold (with or without boundary) in some Euclidean space R^n. We shall say that M^m is *in normal position* in R^n if it is possible to define through each point p of M^m an $(n-m)$-plane $H^{n-m}(p)$ which varies continuously with p and is transversal to M^m at p. Let P^m be a polyhedral m-manifold in R^n. Then we shall say that P^m is *in locally normal position* in R^n if the star of any vertex on P^m is in normal position in R^n. Then we obtain the following:

Theorem 1. *Any polyhedral $(n-1)$-manifold P^{n-1} in locally normal position in the n-dimensional Euclidean space R^n is in normal position.*

Outline of the proof: Let ε be a positive number less than $\frac{1}{n}$. Let s^j be any j-simplex of P^{n-1} and let s^{n-1} be any $(n-1)$-simplex of P^{n-1} which belongs to the star of s^j on P^{n-1}. We choose barycentric coordinates $(u_0, u_1,\cdots, u_{n-1})$ on s^{n-1} so that $u_{j+1}=\cdots=u_{n-1}=0$ at s^j. Let $N_{s^{n-1}}(s^j)$ be the set of points whose barycentric coordinates $(u_0,\cdots, u_{n-1})$ satisfy the following:

$$\varepsilon\leqq u_0,\cdots,\varepsilon\leqq u_j,\ 0\leqq u_{j+1}\leqq\varepsilon,\cdots,\ 0\leqq u_{n-1}\leqq\varepsilon.$$

We shall define

$$N(s^j)=\sum_{s^{n-1}\in St(s^j)} N_{s^{n-1}}(s^j)$$

where $St(s^j)$ is the star of s^j on P^{n-1}.

Thus P^{n-1} is covered by these closed $(n-1)$-dimensional regions $N(s^j)$ which are disjoint from each other except eventually for common faces. We shall define transversal lines on $N(s^j)$ step by step by induction on the dimension of the simplexes of P^{n-1}.

The initial step of induction is to define transversal lines on $N(s^0)$ of any vertex s^0 of P^{n-1}. According to the hypothesis of the theorem, we may define a line $H(s^0)$ which passes through s^0 and is transversal to the star of s^0 at s^0. Then we define a transversal line $H(p)$ through p on $N(s^0)$ by the requirement

$$H(p) \,||\, H(s^0).$$

If transversal lines $H(p)$ are defined on any $N(s^k)$ $(k<j)$, the general step of induction is to extend the definition of $H(p)$ over $N(s^j)$ where s^j is any j-simplex of P^{n-1}. Let t^j be the set of points where all the barycentric coordinates for s^j exceed ε. Then $H(p)$ is already defined on $\overline{s^j-t^j}$ by induction.

First we shall extend the definition of $H(p)$ over t^j. Let $L(t^j)$ be the totality of the lines through the origin of R^n parallel to some $(n-1)$-simplex in the star of s^j on P^{n-1}. Then $L(t^j)$, regarding as a subset of the $(n-1)$-dimensional projective space S^{n-1} composed of all the lines through the origin of R^n, subdivides S^{n-1} in some closed $(n-1)$-dimensional domains $D_i(t^j)$ which are distinct from each other save eventually for common faces. It may be shown that any line $H(p)$ through a point p of t^j is transversal at p to $N(s^j)$ if and only if the line through the origin of R^n parallel to $H(p)$ is a point of the interior $D'_{i_0}(t^j)$ of a fixed domain $D_{i_0}(t^j)$. Thus we obtain a mapping of the boundary of t^j into $D'_{i_0}(t^j)$. By the contractibility of $D'_{i_0}(t^j)$ we may extend this mapping from the t^j into $D'_{i_0}(t^j)$. This is nothing but the constructibility of $H(p)$ on t^j.

Let s^{n-1} be an $(n-1)$-simplex in the star of s^j on P^{n-1}. Let t^{n-1} be the set of points where all barycentric coordinates for s^{n-1} exceed ε. Let t'^j be the bounding simplex of t^{n-1} parallel to t^j. Let t''^{n-j-2} be the bounding simplex of t^{n-1} opposite to t'^j. Consider any point q on t^j. Denote by $B^{n-j-1}_{s^{n-1}}(q)$ the intersection of $N_{s^{n-1}}(s^j)$ with the $(n-j-1)$-dimensional plane determined by q and t''^{n-j-2}, and define $B^{n-j-1}(q)$ as follows:

$$B^{n-j-1}(q)=\sum_{s^{n-1}\in St(s^j)} B^{n-j-1}_{s^{n-1}}(q)$$

where $St(s^j)$ is the star of s^j on P^{n-1}.

As q ranges over t^j, the set $B^{n-j-1}(q)$ fills out $N(s^j)$ in a one-to-one continuous way. If now q is any point of t^j and p is any point of $B^{n-j-1}(q)$, then $H(p)$ will mean the line through p parallel to $H(q)$. This completes the definition of $H(p)$ on $N(s^j)$, and the theorem is proved.

In any arbitrary neighbourhood of a polyhedral m-manifold P^m in normal position in some Euclidean space, there exists, according to

S. S. Cairns [2], an analytic manifold which is homeomorphic to P^m and is an approximation to P^m. Therefore we obtain the following:

Theorem 2. *Under the same condition as Theorem 1, there exists in an arbitrary neighbourhood of P^{n-1} an analytic manifold which is homeomorphic to P^{n-1} and an approximation to P^{n-1}.*

Next we shall say that a topological m-manifold M^m in some Euclidean space R^n is *in regular position* in R^n if there exist unit vectors $v_1(p),\cdots,v_{n-m}(p)$ through each point p of M^m such that $v_1(p),\cdots,v_{n-m}(p)$ vary continuously with p and that the $(n-m)$-plane spanned by these vectors in transversal to M^m at p.

If $(n-1)$-manifold M^{n-1} is in normal position in the n-dimensional Euclidean space R^n, then M^{n-1} is necessarily orientable and divides R^n in two domains D_1 and D_2. We may orient any transversal line defined on M^{n-1} in the direction from the domain D_1 to the domain D_2. Thus we obtain the following:

Theorem 3. *Any $(n-1)$-manifold in normal position in the n-dimensional Euclidean space is in regular position.*

According to H. Whitney [4] any m-manifold M^m in regular position in the n-dimensional Euclidean space R^n may be imbedded in an $(n-m)$-parameter family of analytic manifolds which are homeomorphic to M^m and fill out a neighbourhood of M^m in R^n. Therefore we obtain the following:

Theorem 4. *Under the same condition of Theorem 1, there exists a one parameter analytic family of manifolds M_t ($|t|<1$) which are homeomorphic to P^{n-1} and fill out a neighbourhood of P^{n-1} in R^n and are analytic except for at $t=0$.*

2. Let P^{n-1} be a compact polyhedral $(n-1)$-manifold in regular position in the n-dimensional Euclidean space R^n. We may define through each point p of P^{n-1} a unit vector $v(p)$ which varies continuously with p and transversal to P^{n-1} at p. As each point p of P^{n-1} corresponds to $v(p)$, we obtain a continuous mapping φ of P^{n-1} into a unit sphere S^{n-1}. As P^{n-1} is orientable, we may define the degree of the mapping φ which is independent of $v(p)$ defined on P^{n-1} under the conditions that $v(p)$ varies continuously with p and is transversal to P^{n-1} at p. Then we define the *curvatura integra* $d(P^{n-1})$ of P^{n-1} in R^n as the degree of the mapping φ.

If M^m is an analytic manifold in some Euclidean space R^n, then, according to S. S. Cairns [1], M^m may be so triangulated into cells (σ) that the vertices of each m-cell determine a non singular m-simplex and that the totality of simplexes so determined is a polyhedral manifold P^m homeomorphic to M^m in such a way that corresponding m-cells have identical vertices and that the tangent m-plane to M^m at any point of a cell σ^m of (σ) differs arbitrarily small in its direction from the m-plane of P^m. We shall call P^m a *Cairns' approximation* of M^m in R^n.

Let M^{n-1} be a compact analytic manifold in R^n and let P^{n-1} be a Cairns' approximation of M^{n-1} in R^n. Then constructing at any point p on P^{n-1} the line $H(p)$ parallel to the normal line at the corresponding point of M^{n-1}, it is shown that P^{n-1} is in normal position and the curvatura integra of P^{n-1} in R^n is equal to the usual curvatura integra of M^{n-1} in R^n. Using this fact we obtain the following:

Theorem 5. *If P^{n-1} is a compact polyhedral $(n-1)$-manifold in regular position in R^n and if M_t^{n-1} is the manifold defined in Theorem 4, then the usual curvatura integra of M_t^{n-1} $(t\neq 0)$ in R^n is equal to the curvatura integra of P^{n-1} in R^n.*

Let P^{n-1} and Q^{n-1} be compact polyhedral $(n-1)$-manifolds in R^n. Then we may say that P^{n-1} and Q^{n-1} are *congruent* in R^n, if there exists an orientation preserving semi-linear homeomorphism Ψ of R^n which satisfies $\Psi(P)=Q$. Then there exists, according to V. K. A. M. Gugenheim [3], a piecewise linear homeomorphism $\Phi(p,t)=(\phi_t(p),t)$ of $P^{n-1}\times[0,1]$ into $R^{n-1}\times[0,1]$ such that $\phi_t(p)$ is a peicewise linear homeomorphism of P^{n-1} into R^n.

If P^{n-1} and Q^{n-1} are in regular position in R^n, then we may choose Φ so that $\phi_t(P^{n-1})$ is in regular position in R^n. From this fact we obtain the following:

Theorem 6. *If P^{n-1} and Q^{n-1} are compact polyhedral $(n-1)$-manifolds in R^n and are congruent in R^n, then $d(P^{n-1})=d(Q^{n-1})$.*

References

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