19. On Strictly Continuous Convergence of Continuous Functions

By Takesi ISIWATA
Tokyo University of Education, Tokyo
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1. Let X be a topological space and $C(X)$ be the set of all real-valued continuous functions defined on X. A topology of $C(X)$ is said to be *admissible* provided that $f(x)$ is jointly continuous with respect to the given topologies of X and $C(X)$ respectively. We denote by "$\{f_n\}\to f$ (*jointly*)" that a sequence $\{f_n\}$ converges to f with respect to some admissible topology of $C(X)$. A sequence $\{f_n\}$ is said to be *continuously convergent* to f (abbreviated to $\{f_n\}\to f$ (*cont.*)) if $\{x_n\}\to x$ implies $\{f_n(x_n)\}\to f(x)$. A sequence $\{f_n\}$ is said to be *strictly continuous convergent* to f (abbreviated to $\{f_n\}\to f$ (*str. cont.*) if $\{f(x_n)\}\to\alpha$, then $\{f_n(x_n)\}\to\alpha$ where α is a real number. Finally we shall define "$\{f_n\}\to f$ (*uniformly*)" when a sequence $\{f_n\}$ is uniformly convergent to f. For simplicity, by the property (S), we shall mean the following:

(S): $\{f_n\}\to f$ (*cont.*) *implies* $\{f_n\}\to f$ (*str. cont.*).

Recently, Iséki [1-3] investigated the relations between concepts of (strictly) continuous convergence, pseudo-compactness and countable compactness. In this paper, we shall prove the following:

Theorem 1. *Let X be a countably compact T_1-space. Then $\{f_n\}\to f$ (jointly) if and only if $\{f_n\}\to f$ (str. cont.)* (hence by Theorem 2 in [1], $\{f_n\}\to f$ *(jointly) if and only if* $\{f_n\}\to f$ *(uniformly)*).

Theorem 2. *Let Z be any topological space and X be any dense subset of Z. If X has the property (S), then Z has the property (S).*

The converse of Theorem 2 is not necessarily true (cf. Example 1 below).

Corollary. *Let X be a completely regular T_1-space, and Z be the Čech compactification of X. If X has the property (S), then any subspace Y of Z, $X\subset Y$, has the property (S).*

From Corollary, we can construct a pseudo-compact space which has the property (S) without being countably compact (cf. Example 2 below). Finally, we shall show the existence of a compact space which has not the property (S), by the following

Theorem 3. *Let X be any discrete space containing infinitely many points, and Z be the Čech compactification of X; then we have the following statements:*

i) *Z has no convergent sequence.*

ii) *For any sequence* $\{f_n\}$ *and any function* f *(in* $C(Z)$*), we have* $\{f_n\}\to f$ *(cont.).*

iii) *In* $C(Z)$ *there exists a sequence* $\{f_n\}$ *such that* $\{f_n\}\to f\equiv 0$ *(str. cont.) does not hold.*

In Theorem 3, i) is equivalent to ii) for any completely regular T_1-space Z (cf. Remark 1 below).

2. Proof of Theorem 1. Suppose that $\{f_n\}\to f$ (*jointly*) and there is a sequence $\{x_n; x_n\in X, n=1,2,\cdots\}$ such that $\{f(x_n)\}\to\alpha$ (α being a real number) but not $\{f_n(x_n)\}\to\alpha$. Then there are a subsequence $\{x_{n_i}; i=1,2,\cdots\}$ $(=A)$ of $\{x_n\}$ and some $\varepsilon>0$ such that

$$|f_{n_i}(x_{n_i})-\alpha|\geq\varepsilon. \tag{1}$$

Since X is countably compact, there exists an accumulation point x of A. From the definition of admissible topology, for $\delta>0$ such that $3\delta>\varepsilon$, there are neighborhoods U of f and V of x respectively such that if $U\ni g$, $V\ni y$, then

$$|f(x)-g(y)|<\delta.$$

On the other hand, there is an index n_0 such that $n_0<n_i$ implies $f_{n_i}\in U$, and we have $x_{n_j}\in V$ for some $n_j>n_0$.
Hence

$$|f(x)-f_{n_j}(x_{n_j})|<\delta,$$
$$|f(x)-f(x_{n_j})|<\delta.$$

Therefore we have

$$|f_{n_j}(x_{n_j})-\alpha|\leq|f_{n_j}(x_{n_j})-f(x)|+|f(x)-f(x_{n_j})| + |f(x_{n_j})-\alpha|. \tag{2}$$

Since $\{f(x_n)\}\to\alpha$, we can assume that

$$|f(x_{n_j})-\alpha|<\delta.$$

Hence we have

$$|f_{n_j}(x_{n_j})-\alpha|<3\delta<\varepsilon.$$

This contradicts (1). Thus we have proved that $\{f_n\}\to f$ (*jointly*) implies $\{f_n\}\to f$ (*str. cont.*).

The converse is obvious from the fact that in a pseudo-compact space, $\{f_n\}\to f$ (*str. cont.*) implies $\{f_n\}\to f$ (*uniformly*) (Theorem 2 in [1]).

3. Proof of Theorem 2. Suppose that $\{f_n\}\to f$ (*cont.*), $f(x_n)\to\alpha$ (α=a real number) hold and $\{f_n(x_n)\}\to\alpha$ does not, where $C(Z)\ni f, f_n$ and $Z\ni x_n$ $(n=1,2,\cdots)$. Then there are a subsequence $\{x_{n_i}; i=1,2,\cdots\}$ and $\varepsilon>0$ such that

$$|f_{n_i}(x_{n_i})-\alpha|\geq\varepsilon. \tag{3}$$

Then $\{x_{n_i}\}\frown X$ is a finite set, for if this intersection contains infinitely many points, then the inequality (3) contradicts the property (S) of X. Hence we can assume that $\{x_{n_i}\}\subset Z-X$. We take $\delta>0$ so that $\varepsilon>2\delta>0$. Then there is a neighborhood U_i of x_{n_i} for each i such that if $z\in U_i$ then

(4) $$|f(x_{n_i})-f(z)|<\delta/2^i,$$
(5) $$|f_{n_i}(x_{n_i})-f_{n_i}(z)|<\delta/2^i.$$

Now we choose a point z_i from $U_i \frown X$ for each i. Since $\alpha=\lim_i f(x_{n_i})$, we get

$$\alpha=\lim_i f(x_{n_i})=\lim_i f(z_i)$$

by (4). Since $\{z_i\}\subset X$ and $\{f_n\}\to f$ (*cont.* on Z and *str. cont.* on X), we have

$$\lim_i f(z_i)=\lim_i f_{n_i}(z_i).$$

Therefore, from (5) we have

$$\lim_i f_{n_i}(z_i)=\lim_i f_{n_i}(x_{n_i}).$$

Consequently $\alpha=\lim_i f_{n_i}(x_{n_i})$. This contradicts the inequality (3), hence Z has the property (S).

4. Example 1. We shall construct an example for which the converse of Theorem 2 does not hold. Let us consider, in 2-plane, the following sets:

$$Z=\{(x, y);\ 0\leq x, y\leq 1\},\quad X=Z-(1, 1).$$

Then Z is a compactum and hence Z has the property (S), because Iséki [1] proved that a sequentially compact space has the property (S). But, since X is not pseudo-compact, X has not the property (S) (see Remark 2 below).

The proof of Corollary is obvious.

From Corollary, we shall construct a pseudo-compact space which has the property (S) but is not countably compact.

Example 2. Let Y be a sequentially compact space and $I=[0, 1]$. Then $Y\times I$ is sequentially compact and $\beta(Y\times I)=\beta(Y)\times I$ [4]. There is a completely regular T_1-space X which is pseudo-compact but not countably compact (therefore not normal) such that $Y\times I\subsetneqq X\subsetneqq\beta(Y\times I)$ [5]. Since $Y\times I$ is sequentially compact, $Y\times I$ has the property (S) and hence, by Theorem 2, X has the property (S). Next we shall give a concrete example having the properties mentioned above; let ω and Ω be the least ordinal numbers of the second and third classes respectively. Let $X_0=[1, \Omega]\times[1, \omega]-(\Omega, \omega)$ where a topology of X_0 is given by the order topology; then X_0 has the properties mentioned above.

5. Proof of Theorem 3. i) Let $\{x_n\}$ $(=A)$ be a convergent sequence in Z and $\{x_n\}\to x$. We can assume that $x_n\neq x$ for each n and each x_n is an isolated point in A. Therefore there are open sets U_n in Z $(n=1, 2, \cdots)$ containing x_n such that

$$\overline{U}_n\frown\overline{U}_m=\theta \text{ (the empty set) } (n\neq m),$$

where "—" denotes the closure operation in Z. Let

$$V_n = X \frown \overline{U}_n \quad (n=1, 2, \cdots)$$
$$B = \bigcup_{n=1}^{\infty} V_{2n}, \quad C = \bigcup_{n=1}^{\infty} V_{2n+1}.$$

Then B and C are disjoint closed sets in X and $\overline{B} \frown \overline{C} = \theta$ in Z since X is discrete and hence normal. On the other hand we have $\overline{B} \supset \{x_{2n}\}$ and $\overline{C} \supset \{x_{2n+1}\}$ which contradict that $\{x_n\} \to x$.

ii) Obvious.

iii) Let $\{a_n;\ n=1, 2, \cdots\}$ $(=A) \subset X$. We shall define a function g_n on X for each n in the following way:

$$g_n(a_m) = n/m,$$
$$g_n(z) = 0 \quad \text{for } z \notin A.$$

Then an extension f_n of g_n over Z is identically zero on $Z-X$, because for sufficiently small neighborhood of $x \in Z-X$ contains either a point of $X-A$ or a point a_m with sufficiently large index m, hence $f_n(x)=0$. Let f be the function which is identically zero: $f \equiv 0$. Then we have $f(a_n)=0$ for all n, but $f_n(a_n)=1$ for each n. Therefore $\{f_n\} \to h \equiv 0$ (*str. cont.*) does not hold.

6. Remark 1. In Theorem 3, i) is equivalent to ii) for any completely regular T_1-space. To see this, it is sufficient to prove ii)$\to$i). We suppose that there exists a convergent sequence $\{x_n\} \to x$ $(x_n \neq x;\ n=1, 2, \cdots)$ and $x_n \neq x_m$ $(n \neq m)$. Let $\{U_n;\ n=1, 2, \cdots)$ be a family of neighborhoods of x such that

$$U_n \not\ni x_j, \quad j=1, 2, \cdots, n,$$
$$U_n \ni x_i \quad i=n+1, \cdots.$$

Let f_n be a continuous function such that

$$f_n(y)=0 \quad \text{for } y \in X-U_n$$
$$f_n(x)=1 \quad \text{and} \quad 0 \leq f \leq 1 \text{ on } X.$$

Then $f_n(x_n)=0$, hence $\{f_n\} \to f \equiv 1$ (*cont.*) does not hold.

Remark 2. A sequence $\{f_n\}$ described in (p. 425 in [1], p. 356 in [2] and p. 527 in [3]) is not necessarily continuously convergent to $f \equiv 0$. Such an example is given by the space X_0 described in Example 2. Let $a_n=(\Omega, n)$, then $\{f_n\} \to f \equiv 0$ (*cont.*) does not hold. However, it is true that if a space X has the property (S), then X must be pseudo-compact. For, if X is not pseudo-compact, then there exists a family $\{U_n;\ n=1, 2, \cdots\}$ of open sets such that $\overline{U}_n \frown \overline{U}_m = \theta$ $(n \neq m)$ and $\overline{(\bigcup_{n=1}^{\infty} U_n)} = \bigcup_{n=1}^{\infty} \overline{U}_n$. For each n, we define a continuous function f_n:

$$f_n(y)=0 \quad \text{for } y \notin U_n,$$
$$f_n(x_n)=1 \quad \text{where } x_n \text{ is a fixed point in } U_n,$$
$$0 \leq f \leq 1 \quad \text{on } X.$$

Let $\{y_n\} \to y$ be any convergent sequence. If $y \in \overline{U}_n$ for some n, then $f_m(y_m)=0$ for all $m > n_0$ (n_0 being a suitable integer). If $y \notin \bigcup_{n=1}^{\infty} \overline{U}_n$, then a suitable neighborhood of y is disjoint from $\overline{U_n}$ for each n, and hence we have $f_m(y_m)=0$. Therefore we can conclude that $\{f_n\}\to f\equiv 0$ (*cont.*) but not $\{f_n\}\to f\equiv 0$ (*str. cont.*) because $f_n(x_n)=1$ and $f(x)=0$. From this fact it follows that if X belongs to the class $[N_2]$, then X is countably compact [6].

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