60. *A Characterisation of Compact Metric Spaces*

By Kiyoshi ISÉKI
Kobe University

(Comm. by K. KUNUGI, M.J.A., May 13, 1958)

Recently, B. Grünbaum [3] has given an interesting characterisation of compact metric space, which generalizes a result by A. A. Monteiro and M. M. Peixoto [4]. In this Note, we shall give a characterisation of compact metric spaces by using a property of continuous functions on the metric spaces. Such a property was introduced by M. M. Wainberg [1] to study non-linear operations on Banach spaces.

Let S be a metric space with a metric ρ, and let $f(x)$ be a function defined on the space S. For any two sequences $\{x_n\}$, $\{x'_n\}$ such that $\rho(x_n, x'_n)\to 0$ $(n\to\infty)$,*) we suppose that there are subsequences $\{x_{n_k}\}$ and $\{x'_{n_k}\}$ of the sequences $\{x_n\}$ and $\{x'_n\}$ respectively and, $\lim_{k\to\infty} f(x_{n_k}) =\lim_{k\to\infty} f(x'_{n_k})$ has a finite value. Following M. M. Wainberg, we shall say that such a function $f(x)$ is *completely compact* on S. Then we shall show the following

Theorem. A metric space S is compact, if and only if every continuous function on S is completely compact on S.

Proof. If S is compact, for any two sequences $\{x_n\}$ and $\{x'_n\}$ such that $\lim \rho(x_n, x'_n)=0$, we can find convergent subsequences $\{x_{n_k}\}$ and $\{x'_{n_k}\}$ having a limit point x_0 in S. Therefore we have $\lim_{k\to\infty} f(x_{n_k})=\lim_{k\to\infty} f(x'_{n_k}) =f(x_0)$.

Conversely, we shall suppose that every continuous function on S is completely compact. Let S be a non-compact metric space, and let $A=\{x_1, x_2,\cdots, x_n,\cdots\}$ be any countably infinite subset of S having no limit point. Then the set A is closed, and the function $g(x)$ on A such that $g(x)=n$ $(n=1, 2,\cdots)$ is continuous on A. By Tietze theorem, we can extend $g(x)$ over the space S continuously. Let $f(x)$ be its extended function, then $f(x)$ is not completely compact. For, if $x_n=x'_n$ $(n=1, 2,\cdots)$, $\lim_{k\to\infty} \rho(x_n, x'_n)=0$, and for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$, we have $\lim_{k\to\infty} f(x_{n_k})=+\infty$.

References

[1] M. M. Wainberg (М. М. Вайнберг): Variation Method of Study of Non-linear Operators, Gostekhizdat, Moscow (Вариационные Методы Исследования Нелинейных Операторов, Гостехиздат, Москва) (1956).

*) In his paper [2], R. Doss has given the condition that a metric space has Lebesgue property by the notion of such an accessible sequence.

[2] R. Doss: On uniformly continuous functions in metrizable spaces, Proc. Math. Soc. Egypt, **3**, 1-6 (1947).
[3] B. Grünbaum: A characterization of compact metric spaces, Riveon Lematematika, **9**, 70-71 (1955).
[4] A. A. Monteiro and M. M. Peixoto: Le nombre de Lebesgue et continuité uniforme, Portugaliae Math., **10**, 105-113 (1951).