# 56. On Homomorphic Mappings

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In the theory of real valued functions we have

"Theorem A. Let $R$ be the space of real numbers and $f(x)$ an additive function defined on $R$. If $f(x)$ is measurable (with respect to the Lebesgue measure), then $f(x)$ is continuous".

This is a well-known theorem. It will be natural to propose the following question, in connection with the above theorem:

"Let $G$ and $G^*$ be two topological groups and $\varphi(x)$ a homomorphic mapping of an abstract group $G$ into an abstract group $G^*$. Under what conditions does it follow that $\varphi(x)$ is a continuous mapping of the topological group $G$ into the topological group $G^*$"?

It is the purpose of the present paper to answer this question. First we shall extend Theorem A to a more general case (see Theorem 1). This generalization is the first answer for the above question. Next we shall prove a theorem (Theorem 2) which is the second answer for the above question. And we have, using our Theorems 1 and 2 and the duality theorem of Pontrjagin, an interesting consequence (see Theorem 3).

Definition 1. Let $G$ be an abstract space and $m^*$ an outer measure in $G$. Let $f$ be a mapping of $G$ into a topological space $\Omega$. $f$ is called an $m^*$-measurable mapping if the set $f^{-1}(U)$ is $m^*$-measurable for every open set $U\subseteq\Omega$.

Definition 2. Let $G$ be a topological space. Let $f$ be a mapping of $G$ into a topological space $\Omega$. $f$ is called a mapping which has the property of Baire if the set $f^{-1}(U)$ has the property of Baire for every open set $U\subseteq\Omega$.

Definition 3. Let $G$ be a topological group. $G$ is called to be $\sigma$-bounded, if for every open set $U\subseteq G$ there exists a sequence $a_1, a_2,\cdots,a_n,\cdots$ of elements of $G$ such that $G=\bigcup_{i=1}^{\infty}a_iU$.

Theorem 1. Let $G$ be a locally compact group and $m^*$ a left-invariant Haar's outer measure in $G$. If $f$ is an $m^*$-measurable homomorphic mapping of $G$ into a $\sigma$-bounded topological group $G^*$, then $f$ is continuous.

Proof. Let $H^*=f(G)$. If we introduce the relative topology in $H^*$, then $H^*$ becomes a $\sigma$-bounded topological group. For the proof of our theorem it is sufficient to show that $f$ is a continuous mapping of $G$ into $H^*$. Let $U^*$ be an arbitrary neighborhood of the identity

$e^*$ of $H^*$. There exists a neighborhood $V^*$ of $e^*$ such that $V^{*-1}V^* \subseteq U^*$. Let $V=f^{-1}(V^*)$. From Definition 1 we see that $V$ is $m^*$-measurable. We shall show that $m(V)>0$. There exists a sequence $a_1^*, a_2^*, \cdots, a_n^*, \cdots$ of elements of $H^*$ such that $H^* = \bigcup_{i=1}^{\infty} a_i^* V^*$. We set $V_i = f^{-1}(a_i^* V^*)$, $i=1, 2, \cdots$ . Then it is easily seen that each $V_i$ is written in the form $a_i V$, where $a_i$ is an arbitrary element of $f^{-1}(a_i^*)$. Hence we have $G = \bigcup_{i=1}^{\infty} a_i V$. From this we can easily see that $m(V)>0$. There exists a neighborhood $W$ of the identity $e$ of $G$ such that $W \subseteq V^{-1}V$ (this is the well-known fact in the theory of Haar's measure). Thus we have $f(W) \subseteq f(V^{-1}V) \subseteq V^{*-1}V^* \subseteq U^*$. This shows that $f$ is continuous at $e$. On the other hand $f$ is a homomorphic mapping of an abstract group $G$ onto an abstract group $H^*$. Hence $f$ is continuous at all points.

Corollary. Let $R$ be the space of real numbers. And let $f(x)$ be a real-valued function defined on $R$ such that $f(x+y)=f(x)+f(y)$. If $f(x)$ is a Lebesgue-measurable function, then $f(x)$ can be written in the form $f(x)=\lambda x$.

Lemma 1. Let $G$ be a topological group whose open sets are all of the second category. And let $M \subseteq G$ be a subset which has the property of Baire. If $M$ is of the second category, then $M^{-1}M$ contains a neighborhood $V$ of the identity $e$ of $G$.

Proof. Since $M$ has the property of Baire, there exists an open set $U$ such that the symmetric difference $M \ominus U$ is of the first category. On the other hand $M$ is of the second category, and hence we can easily see that $U \neq 0$. We take an arbitrary element $a$ of $U$. There exists a neighborhood $V$ of the identity $e$ such that

$$VV^{-1} \subseteq a^{-1}U. \tag{1}$$

We set $K = M \ominus U$. Then we have

$$V \subseteq VV^{-1} \subseteq a^{-1}U \subseteq a^{-1}M \cup a^{-1}K. \tag{2}$$

Let $b$ be an arbitrary element of $V$. From (1) we have

$$Vb^{-1} \subseteq a^{-1}U \subseteq a^{-1}M \cup a^{-1}K, \text{ that is, } V \subseteq a^{-1}Mb \cup a^{-1}Kb. \tag{3}$$

Since $V$ is of the second category and both $a^{-1}K$ and $a^{-1}Kb$ are of the first category, it is evident that (using (2) and (3))

$$(a^{-1}M \cap a^{-1}Mb) \cap V \neq 0, \text{ that is, } M \cap Mb \neq 0. \tag{4}$$

This implies that for an arbitrary element $b \in V$ there exist elements $c \in M$ and $d \in M$ such that $c=db$, that is, $d^{-1}c=b$. Hence we have $M^{-1}M \supseteq V$.

By using Lemma 1, we can also prove Theorem 2 below.

Theorem 2. Let $G$ be a topological group whose open sets are all of the second category. And let $f$ be a homomorphic mapping of $G$ into a $\sigma$-bounded topological group $G^*$. If $f$ has the property of Baire, then $f$ is continuous.

Theorem 3. Let $G$ be a separable and locally compact abelian group. Suppose that $G$ is not discrete. Then there exists at least one set $E_1 \subseteq G$ which is not measurable with respect to the Haar measure in $G$. And further there exists at least one set $E_2 \subseteq G$ which does not have the property of Baire.

Proof. We introduce the discrete topology in $G$ and denote this topological group by $G^*$. Let $X$ and $X^*$ be the character groups of $G$ and $G^*$ respectively. Then $X$ is a separable and locally compact abelian group and $X^*$ is a compact abelian group. Clearly an element $\chi \in X$ can be regarded as an element $\chi^* \in X^*$. To every $\chi \in X$ we correspond such an element $\chi^* \in X^*$. Then we have a mapping $\psi(\chi) = \chi^*$ of $X$ into $X^*$. It is easily seen that $\psi(\chi)$ is a continuous homomorphic mapping of the topological group $X$ into the topological group $X^*$. We shall show that $\psi(X) \neq X^*$. Suppose that $\psi(X) = X^*$. Then $\psi^{-1}$ is also continuous. (This is a well-known fact in the theory of topological groups.) Hence $X$ is homeomorphic with $X^*$ and consequently a compact group. This implies that $G$ is discrete. (Remember the duality theorem of Pontrjagin.) Thus we have arrived at a contradiction. Hence there exists a $\chi^* \in X^*$ which does not belong to $\psi(X)$. Clearly $\chi^*$ is a homomorphic mapping of an abstract group $G$ into an abstract group $K$ ($K$ is the factor group $R/N$, where $R$ is the additive topological group of real numbers and $N$ is the subgroup of all integers). But this is not a continuous mapping of $G$ into $K$. Hence by Theorem 1 $\chi^*$ is not a measurable mapping (with respect to the Haar measure in $G$) of $G$ into $K$, and by Theorem 2 $\chi^*$ is not a mapping which has the property of Baire. Consequently $\chi^{*-1}(U)$ is non-measurable for a certain open set $U \subseteq K$ and $\chi^{*-1}(V)$ is a set which does not have the property of Baire for a certain open set $V \subseteq K$. Setting $E_1 = \chi^{*-1}(U)$ and $E_2 = \chi^{*-1}(V)$, we obtain our theorem.

Lemma 2. Let $R$ be the space of real numbers. Then there exists a set $B$ with the properties:

1) For every $x \in R$ there exists a finite subset $\{x_1, x_2, \cdots, x_n\}$ of $B$ and a corresponding finite set $\{r_1, r_2, \cdots, r_n\}$ of rational numbers such that $x = \sum_{i=1}^{n} r_i x_i$.

2) $B$ is linearly independent with respect to rational coefficients, that is, $r_1x_1 + r_2x_2 + \cdots + r_nx_n = 0$ implies $r_1 = r_2 = \cdots = r_n = 0$ for every finite subset $\{x_1, x_2, \cdots, x_n\}$ of $B$ and a finite set $\{r_1, r_1, \cdots, r_n\}$ of rational numbers.

This is well known. $B$ is called a Hamel basis. It is easily proved that every linearly independent set (in the sense of rational coefficient) is contained in a Hamel basis.

Example. Let $R$ be the space of real numbers. There exists a subset $G \subseteq R$ satisfying the following three conditions:

1) $G$ is an abstract subgroup of $R$.

2) $G$ is a non-measurable (in the sense of Lebesgue) set of $R$.

3) $G$ is a set which does not have the property of Baire.

Proof. Let $B$ be a Hamel basis containing 1. Let $H$ be the subgroup of the rational numbers and $G$ the subgroup which is generated by the rational linear combinations of elements of $B-\{1\}$. Then it is easily seen that $R$ is decomposed into the direct sum of $H$ and $G$. Hence every element $x \in R$ is written in the form $x=h+g$, where $h \in H$ and $g \in G$. We define $f(x)=g$, for $x=h+g$, $h \in H$, $g \in G$. Clearly $f(x)$ is a homomorphic mapping of $R$ into itself. It is not hard to show that $f(x)$ is not continuous. Hence by Theorem 1 $f(x)$ is not measurable. And by Theorem 2 $f(x)$ is not a function which has the property of Baire. Then we can easily prove that $G$ satisfies the above conditions 2) and 3). (Notice that $\overline{\overline{H}} = \aleph_0$.)