77. On Convergence Criteria for Fourier Series. I

By Kenji YANO

Department of Mathematics, Nara Women's University, Nara, Japan

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1. Introduction. Let $\varphi(t)$ be an even function, integrable in $(0, \pi)$, periodic of period 2π, and let

$$\varphi(t)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nt$$

and

$$s_n=\frac{1}{2}a_0+\sum_{\nu=1}^{n}a_\nu.$$

Hardy and Littlewood [1] proved the following

THEOREM A. If

$$\int_0^t|\varphi(u)-s|\,du=o\left(t/\log\frac{1}{t}\right)\qquad (t\to 0), \tag{1.1}$$

and if for some positive δ

$$a_n>-An^{-\delta},\ A>0, \tag{1.2}$$

then $s_n\to s$.

The proof requires a very difficult Tauberian theorem, and so later Szász [2] gave an alternate proof under the additional condition

$$|\varphi(t)-s|<t^{-c},\ c \text{ a positive constant.}$$

Recently, Wang [3] and Sunouchi [4] proved Theorem A by the method of Riesz summability, and the latter's extension is as follows:

THEOREM B (Sunouchi). If

$$\int_0^t|\varphi(u)-s|\,du=o\left(t/f\left(\frac{1}{t}\right)\right)\qquad (t\to 0),$$

and if $a_n>-\mu(n, A)$ for some positive A, then $s_n\to s$, where $f(x)$ and $\mu(x, A)$ are defined by the conditions 1° $f(x)>0$, $f'(x)>0$, 2° $F(x)=\int^x(1/uf(u))\,du\uparrow\infty$ as $x\uparrow\infty$, and 3° $\mu(x, A)=1/F^{-1}(F(x)-A)$.

In this paper, we shall first give an another proof to Theorem A by the method of de la Vallée Poussin summability, and generalize it in alternate form slightly different from Theorem B. In § 3 we refer to jump functions.

THEOREM 1. If (1.1) holds, and if for some positive δ

$$s_{n+\nu}-s_n>-\varepsilon_n\quad\text{for}\quad \nu=1,2,\cdots,[n^\delta], \tag{1.3}$$

where $\varepsilon_n>0$ and $\varepsilon_n\to 0$, then $s_n\to s$.

Observing that $\varepsilon_n\to 0$ may be as slowly as we wish, Theorem A is a corollary of Theorem 1 since (1.3) holds whenever $a_n>-An^{-\delta-\varepsilon}$, $\varepsilon>0$, which is (1.2) replaced δ by $\delta+\varepsilon$.

Definition 1. We define $g(x)$ such as 1° $g(x)>0$ for $x\geqq x_0>0$, 2° $g(x)\uparrow\infty$ as $x\uparrow\infty$, and 3° $H\leqq g(x^\delta)/g(x)\leqq 1$, $0<\delta<1$, for all $x\geqq x_0$, where $H=H(\delta)$ is a positive constant depending on δ only.

We may take for $g(x)$, e.g. $(\log x)^\alpha(\alpha>0)$, $\log x\log\log x$ and $\log_p x$, where $\log_p$ is the p-times iterated logarithm. For the sake of simplicity we denote $(g(x))^\alpha$ by $g(x)^\alpha$ throughout this paper.

THEOREM 2. If

$$\int_0^t|\varphi(u)-s|\,du=o\left(t/\log g\left(\frac{1}{t}\right)\right)\qquad (t\to 0),\tag{1.4}$$

where $g(x)$ is defined by Definition 1, and if for some positive $\varDelta$

$$s_{n+\nu}-s_n>-\varepsilon_n\ \text{ for } \nu=1,2,\cdots,[n/g(n)^\varDelta],\quad (n\geqq n_0),^{*)}\tag{1.5}$$

where $\varepsilon_n>0$ and $\varepsilon_n\to 0$, then $s_n\to s$.

COROLLARY 2.1. If (1.4) holds, and if for some positive $\varDelta$

$$a_n>-An^{-1}g(n)^\varDelta,\ A>0,\qquad (n\geqq n_0),\tag{1.6}$$

then $s_n\to s$.

Again observing that ε_n may tend to zero arbitrarily, this follows immediately from Theorem 2 since (1.6) implies (1.5) replaced $\varDelta$ by $\varDelta+1$. This corollary will correspond to Theorem B.

COROLLARY 2.2. Let p be a positive integer. If

$$\int_0^t|\varphi(u)-s|\,du=o\left(t/\log_{p+1}\frac{1}{t}\right)\qquad (t\to 0),$$

and if $a_n>-An^{-1}(\log_p n)^\varDelta$, $n\geqq n_0$, for some positive $\varDelta$, then $s_n\to s$.

This follows from Corollary 2.1 by letting $g(x)=\log_p x$.

2. Proof of Theorems 1 and 2. We need some lemmas. Let $m<n$, then we have the two identities:

$$s_n=\frac{1}{m}\sum_{\nu=1}^m s_{n+\nu}-\frac{1}{m}\sum_{\nu=1}^m(s_{n+\nu}-s_n),\tag{2.1}$$

$$s_n=\frac{1}{m}\sum_{\nu=1}^m s_{n-\nu}+\frac{1}{m}\sum_{\nu=1}^m(s_n-s_{n-\nu}).\tag{2.2}$$

Here we suppose that $m=m(n)$ tends to infinity with n, and is as same order as or lower order than n. If

$$\lim_{n\to\infty}\frac{1}{m}\sum_{\nu=1}^m s_{n+\nu}=\lim_{n\to\infty}\frac{1}{m}\sum_{\nu=1}^m s_{n-\nu}=s,$$

and if $s_{n+\nu}-s_n>-\varepsilon_n$ for $\nu=1,2,\cdots,m$, where $\varepsilon_n>0$ and $\varepsilon_n\to 0$, then from (2.1) and (2.2) we have

$$\limsup_{n\to\infty}s_n\leqq s\quad\text{and}\quad\liminf_{n\to\infty}s_n\geqq s$$

respectively, and then $\lim s_n=s$.

On the other hand, denoting by $D_n(t)$ the Dirichlet kernel,

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*) Essentially n_0 may be expressed by x_0 in Definition 1, and is an absolute constant depending on the function $g(x)$ only.

$$s_{n+\nu}=\frac{2}{\pi}\int_0^\pi \varphi(t)D_{n+\nu}(t)\,dt=\frac{2}{\pi}\int_0^\pi \varphi(t)\frac{\sin(n+1/2+\nu)t}{2\sin(t/2)}\,dt,$$

and so we have

$$\frac{1}{m}\sum_{\nu=1}^{m} s_{n+\nu}=\frac{2}{\pi}\int_0^\pi \varphi(t)\,\frac{2\sin(mt/2)}{m(2\sin(t/2))^2}\sin\Big(n+\frac{1}{2}+\frac{1}{2}(m+1)\Big)t\,dt.$$

It is analogous to the mean $m^{-1}\sum_{\nu=1}^{m} s_{n-\nu}$.

Hence we get the following

LEMMA 1. Let $m=m(n)<n$ tend to infinity with n as same order as or lower order than n. If

$$\frac{2}{\pi}\int_0^\pi \varphi(t)\chi_n(t)\,dt=s+o(1) \qquad (n\to\infty),$$

where

$$\chi_n(t)=\chi_n^{\pm}(m,t)=\frac{2\sin(mt/2)}{m(2\sin(t/2))^2}\sin\Big(n+\frac{1}{2}\pm\frac{1}{2}(m+1)\Big)t \tag{2.3}$$

and if

$$s_{n+\nu}-s_n>-\varepsilon_n \quad \text{for} \quad \nu=1,2,\cdots,m, \tag{2.4}$$

where $\varepsilon_n>0$ and $\varepsilon_n\to 0$, then $s_n\to s$.

Clearly, in this lemma the condition (2.4) may be replaced by

$$a_n>-\varepsilon_n/m. \tag{2.5}$$

LEMMA 2. The kernel $\chi_n(t)=\chi_n^{\pm}(m,t)$ defined by (2.3), m being lower order than n, has the following properties:

$$\frac{2}{\pi}\int_0^\pi \chi_n(t)\,dt=1, \tag{2.6}$$

$$\chi_n(t)=\begin{cases} O(n) & (0\leqq t\leqq\pi)\\ O(1/t) & (nt\geqq 1)\\ O(1/mt^2) & (mt\geqq 1).\end{cases} \tag{2.7}$$

Indeed (2.6) follows from

$$\chi_n(t)=\frac{1}{m}\sum_{\nu=1}^{m} D_{n\pm\nu}(t) \quad\text{and}\quad \frac{2}{\pi}\int_0^\pi D_n(t)\,dt=1,$$

and (2.7) does from the expression of $\chi_n(t)$ in (2.3).

Proof of Theorems 1 and 2. By Lemma 1 it is sufficient to show that

$$I\equiv\int_0^\pi \varphi(t)\chi_n(t)\,dt=\frac{\pi}{2}s+o(1) \qquad (n\to\infty), \tag{2.8}$$

where $\chi_n(t)$ is defined by (2.3). And we may suppose that $s=0$ with no loss of generality by (2.6). We divide I into three parts

$$I=\int_0^{n^{-1}}+\int_{n^{-1}}^{m^{-1}}+\int_{m^{-1}}^{\pi}=I_1+I_2+I_3$$

say, where we put $m=[n^\delta]$, $0<\delta<1$, in Theorem 1, and $m=[n/g(n)^\Delta]$ in Theorem 2. The assumption (1.1) in Theorem 1 or (1.4) in Theorem 2 implies $\int_0^t|\varphi(u)|\,du=o(t)$. And since $\chi_n(t)=O(n)$ by (2.7),

$$I_1=\int_0^{n^{-1}}\varphi(t)\,\chi_n(t)\,dt=O\Big(n\int_0^{n^{-1}}|\varphi(t)|\,dt\Big)=o(nn^{-1})=o(1).$$

Further, since $\chi_n(t)=O(1/mt^2)$ for $mt\geqq 1$ by (2.7),

$$I_3=\int_{m^{-1}}^{\pi}\varphi(t)\,\chi_n(t)\,dt=O\Big(\frac{1}{m}\int_{m^{-1}}^{\pi}\frac{|\varphi(t)|}{t^2}dt\Big)=o(1).$$

Next, using $\chi_n(t)=O(1/t)$ for $nt\geqq 1$ in (2.7),

$$I_2=\int_{n^{-1}}^{m^{-1}}\varphi(t)\,\chi_n(t)\,dt=O\Big(\int_{n^{-1}}^{m^{-1}}\frac{|\varphi(t)|}{t}dt\Big). \tag{2.9}$$

Concerning Theorem 1, the condition (1.1) is written as $\int_0^t|\varphi(u)|\,du=o(t/\log t^{-1})$, and so applying integration by parts to the right hand side integral in (2.9)

$$\begin{aligned}I_2&=\Big[o\Big(1/\log\frac{1}{t}\Big)\Big]_{n^{-1}}^{m^{-1}}+o\Big(\int_{n^{-1}}^{m^{-1}}\Big(1/t\log\frac{1}{t}\Big)dt\Big)\\&=o(1)+o\Big[-\log\log\frac{1}{t}\Big]_{n^{-1}}^{m^{-1}}\\&=o(1)+o\Big(\log\frac{\log n}{\log m}\Big),\quad m=[n^{\delta}],\\&=o(1)+o\Big(\log\frac{1}{\delta}\Big)=o(1).\end{aligned}$$

Concerning Theorem 2, the condition (1.4) is written as $\int_0^t|\varphi(u)|\,du=o(t/\log g(t^{-1}))$, and so as above

$$\begin{aligned}I_2&=\Big[o\Big(1/\log g\Big(\frac{1}{t}\Big)\Big)\Big]_{n^{-1}}^{m^{-1}}+o\Big(\int_{n^{-1}}^{m^{-1}}\Big(1/t\log g\Big(\frac{1}{t}\Big)\Big)dt\Big)\\&=o(1)+o\Big((\log g(m))^{-1}\int_{n^{-1}}^{m^{-1}}\frac{dt}{t}\Big)\\&=o(1)+o\Big((\log g(n))^{-1}\log\frac{n}{m}\Big),\quad m=[n/g(n)^{\Delta}],\\&=o(1)+o((\log g(n))^{-1}\Delta\cdot\log g(n))=o(1).\end{aligned}$$

Thus I_1, I_2 and I_3 are all $o(1)$, and we get (2.8) with $s=0$, which proves both the theorems.

3. Jump functions. Let $\psi(t)$ be an odd function, integrable in $(0,\pi)$, periodic of period 2π, and let

$$\psi(t)\sim\sum_{n=1}^{\infty}b_n\sin nt\quad\text{and}\quad t_n=\sum_{\nu=1}^{n}\nu b_\nu.$$

Then we have the following theorems:

THEOREM 3. If there exists a number l such that

$$\int_0^t|\psi(u)-l|\,du=o\Big(t/\log\frac{1}{t}\Big)\qquad(t\to 0), \tag{3.1}$$

and if for some positive δ

$$n^{-1}(t_{n+\nu}-t_n)>-\varepsilon_n \quad \text{for} \quad \nu=1,2,\cdots,[n^{\delta}],$$

where $\varepsilon_n>0$ and $\varepsilon_n\to 0$, then the sequence $\{nb_n\}$ is summable $(C,1)$ to $2l/\pi$.

COROLLARY 3. If (3.1) holds and if for some positive δ,

$$b_n>-An^{-\delta},\ A>0,$$

then the sequence $\{nb_n\}$ is summable $(C,1)$ to $2l/\pi$.

This follows from Theorem 3 like as Theorem A does from Theorem 1. This corollary is a generalization of a theorem due to Mohanty-Nanda [5] in which the latter condition is replaced by $b_n=O(n^{-\delta})$.

THEOREM 4. If there exists a number l such that

$$\int_0^t |\psi(u)-l|\,du=o\left(t/\log g\left(\frac{1}{t}\right)\right) \qquad (t\to 0), \tag{3.2}$$

where $g(x)$ is defined by Definition 1 in §1, and if for some positive Δ

$$n^{-1}(t_{n+\nu}-t_n)>-\varepsilon_n \quad \text{for} \quad \nu=1,2,\cdots,[n/g(n)^{\Delta}], \qquad (n\geqq n_0),$$

where $\varepsilon_n>0$ and $\varepsilon_n\to 0$, then the sequence $\{nb_n\}$ is summable $(C,1)$ to $2l/\pi$.

COROLLARY 4. If (3.2) holds, and if for some positive Δ

$$b_n>-An^{-1}g(n)^{\Delta},\ A>0, \qquad (n\geqq n_0),$$

then $\{nb_n\}$ is summable $(C,1)$ to $2l/\pi$.

Proof of Theorems 3 and 4. Using the identities

$$\frac{t_n}{n+1}=\frac{1}{(n+1)m}\sum_{\nu=1}^{m}t_{n+\nu}-\frac{1}{(n+1)m}\sum_{\nu=1}^{m}(t_{n+\nu}-t_n)$$

and

$$\frac{t_n}{n+1}=\frac{1}{(n+1)m}\sum_{\nu=1}^{m}t_{n-\nu}+\frac{1}{(n+1)m}\sum_{\nu=1}^{m}(t_n-t_{n-\nu}),$$

where $m=[n^{\delta}]$, $0<\delta<1$, or $m=[n/g(n)^{\Delta}]$, it is sufficient to show that

$$I_n\equiv\frac{1}{(n+1)m}\sum_{\nu=1}^{m}t_{n\pm\nu}=\frac{2l}{\pi}+o(1) \qquad (n\to\infty), \tag{3.3}$$

by Lemma 1. On the other hand, since

$$\nu b_\nu=\frac{2}{\pi}\int_0^{\pi}\psi(t)\,\nu\sin\nu t\,dt=-\frac{2}{\pi}\int_0^{\pi}\psi(t)\frac{d}{dt}\cos\nu t\,dt,$$

t_n is written as

$$t_n=\sum_{\nu=1}^{n}\nu b_\nu=-\frac{2}{\pi}\int_0^{\pi}\psi(t)\frac{d}{dt}D_n(t)\,dt.$$

Substituting this into the expression of I_n in (3.3) we have

$$I_n=-\frac{2}{\pi}\int_0^{\pi}\psi(t)\frac{1}{n+1}\frac{d}{dt}\chi_n(t)\,dt, \tag{3.4}$$

where $\chi_n(t)=\chi_n^{\pm}(m,t)$ coincides with that in (2.3), i.e.

$$\chi_n(t)=\frac{2\sin(mt/2)}{m(2\sin(t/2))^2}\sin\left(n+\frac{1}{2}\pm\frac{1}{2}(m+1)\right)t. \tag{3.5}$$

And, $(n+1)^{-1}(d/dt)\chi_n(t)$ has all the properties of $\chi_n(t)$ in (2.7), i.e.

$$\frac{1}{n+1}\frac{d}{dt}\chi_n(t)=\begin{cases} O(n) & (0\leqq t\leqq \pi) \\ O(1/t) & (nt\geqq 1) \\ O(1/mt^2) & (mt\geqq 1). \end{cases}$$

Now, I_n in (3.4) is

$$\begin{aligned} I_n=&-\frac{2}{\pi}\int_0^\pi (\psi(u)-l)\frac{1}{n+1}\frac{d}{dt}\chi_n(t)\,dt \\ &-\frac{2l}{\pi}\int_0^\pi \frac{1}{n+1}\frac{d}{dt}\chi_n(t)\,dt=K_1+K_2, \end{aligned} \tag{3.6}$$

say. Then, we see that $K_1=o(1)$ under the respective conditions in Theorems 3 and 4 quite analogously as the proof of Theorems 1 and 2. Concerning K_2

$$\begin{aligned} K_2&=-\frac{2l}{\pi}\frac{1}{n+1}[\chi_n(t)]_{t=0}^{\pi}=o(1)+\frac{2l}{\pi}\frac{1}{n+1}\chi_n(0) \\ &=o(1)+\frac{2l}{\pi}\frac{1}{n+1}\left(n+\frac{1}{2}\pm\frac{1}{2}(m+1)\right) \qquad \text{by (3.5),} \\ &=\frac{2l}{\pi}+o(1). \end{aligned}$$

Hence, (3.3) follows from (3.6), and we get the desired results.

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