95. *Some Expectations in AW*-algebras*

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1. Let A be a commutative AW^*-algebra (cf. [2]). We denote by B and P the totality of self-adjoint elements and projections in A, respectively. It is well known that A is isometrically isomorphic to the space $C(S)$ of all complex-valued continuous functions on a Stonean space S. In this representation, B (or P) is the totality of real-valued (or characteristic) functions in $C(S)$ which forms a conditionally complete vector lattice (or complete lattice) by the usual ordering in $C(S)$.

Let M be a left module over B. We shall call a mapping n of M into B an *n-mapping* on M if n satisfies

(1) $\quad n(x+y)\leq n(x)+n(y) \qquad (x, y\in M)$,

(2) $\quad n(ax)=an(x) \qquad (x\in M, a\in A \text{ with } a\geq 0)$.

If a mapping f of a subset $D(f)$ of M into B satisfies

(3) $\quad -n(-x)\leq f(x)\leq n(x)$,

then we call f to be *n-bounded*. In the case when f is additive and when $D(f)$ is an additive subgroup of M, we can replace (3) by the inequality: $f(x)\leq n(x)$.

2. For convenience, we state a simple lemma which is easily verified.

Lemma 1. *Let M be a left module over (not necessarily commutative) AW^*-algebra L and $P(x)$ be a proposition concerning the element x in M. Suppose that the following two conditions are satisfied:*

(4) *If there exists a family $(e_i; i\in I)$ of orthogonal projections in L with l.u.b. 1 such that all $P(e_ix)$ are true, then $P(x)$ is true.*

(5) *For any projection e in L which is not zero, we can find a non-zero projection e' in L such that $e'\leq e$ and $P(e'x)$ is true. Then $P(x)$ is true.*

3. Now we state an extension theorem of Hahn-Banach type.

Theorem 1. *Let M be a left module over B with n-mapping n. Given an n-bounded B-module homomorphism of a B-submodule of M into B, it can be extended to an n-bounded B-module homomorphism of M into B.*

Proof. Let h be an n-bounded B-module homomorphism of a submodule $D(h)$ of M. Let R be the set of all couples $(f, D(f))$, where f is an n-bounded B-module homomorphism of a submodule $D(f)$ of M containing $D(h)$ into B such that $f=h$ on $D(h)$. If we define

$(f_1, D(f_1)) \geq (f_2, D(f_2))$ by the relation that $D(f_1) \supseteq D(f_2)$ and $f_1 = f_2$ on $D(f_2)$, then R is inductively ordered and by Zorn's lemma there exists a maximal element $(f_0, D(f_0))$.

We shall prove that $D(f_0) = M$. Contrary to the assertion, suppose the existence of a non-zero element x_0' in $M - D(f_0)$. Then we can find a non-zero e in P satisfying the condition

(6) for every non-zero $e' \in P$ with $e' \leq e$, we have $e'x_0' \notin D(f_0)$. For, otherwise, taking as $P(x)$, the proposition that x is in $D(f_0)$, we see that $P(x')$ satisfies (5). Moreover $P(x)$ satisfies (4) for all x in M. In fact, if there is an orthogonal family $(e_i;\, i \in I)$ with l.u.b. 1 such that $e_i x \in D(f_0)$ for all i, then we can define

$$g_0(y) = \textstyle\sum_I e_i f_0(e_i y) \qquad (y \in D(g_0) = Bx + D(f_0)),$$

where the right side denotes the unique element $w \in B$ such that $e_j w = e_j f_0(e_j y)$ for all j (cf. Kaplansky [3]). It is easy to show that $(g_0, D(g_0)) \in R$ and $(g_0, D(g_0)) \geq (f_0, D(f_0))$. By the maximality of $(f_0, D(f_0))$, we have $D(g_0) = D(f_0)$; hence $P(x)$ is true. Thus, by Lemma 1, $P(x_0')$ is true; that is, $x_0' \in D(f_0)$ which is a contradiction.

Put $x_0 = ex_0'$, then we have

(6′) $x_0 = ex_0$ and $e'x_0 \notin D(f_0)$ for any non-zero e' in eP. For any $x_1, x_2 \in D(f_0)$, using (1) and (3), we have

$$f_0(x_1) - n(x_1 - x_0) \leq n(x_2 + x_0) - f(x_2).$$

By the conditionally completeness of B, we can find $d' \in B$ such that

$$f_0(x) - n(x - x_0) \leq d' \leq n(x + x_0) - f_0(x) \quad \text{for any } x \in D(f_0).$$

Putting $d = ed'$, we have $e(f_0(x) - n(x - x_0)) \leq d \leq e(n(x + x_0) - f_0(x))$. On the other hand

$$(1-e)(f_0(x) - n(x - x_0)) = f_0((1-e)x) - n((1-e)x) \leq 0,$$
$$(1-e)(n(x + x_0) - f_0(x)) = n((1-e)x) - f_0((1-e)x) \geq 0.$$

From these three inequalities, we finally get

(7) $f_0(x) - n(x - x_0) \leq d \leq n(x + x_0) - f_0(x)$ and $ed = d$.

Denote $D(h_0) = Bx_0 + D(f_0)$. Then $ax_0 + x = 0$ ($a \in B$, $x \in D(f_0)$) implies $ae = 0$ and $x = 0$. To show this we may assume $ae = a$. If $a = 0$, then we can find $e_1 \in P$ with $e_1 \leq e$ such that $(1 - e_1) + ae_1$ has inverse. Then $e_1 x_0 = -((1 - e_1) + ae_1)^{-1} x \in D(f_0)$, which contradicts (6′).

Thus we can define uniquely

$$h_0(y) = ad + f_0(x) \quad \text{for } y = ax_0 + x \in D(h_0).$$

It is easy to verify that h_0 is a B-module homomorphism of $D(h_0)$ into B.

Finally we shall prove that h_0 is n-bounded. Let $P(x)$ be the proposition that $h_0(y) \leq n(y)$ for y in $D(h_0)$. If there exists an orthogonal family $(e_i;\, i \in I)$ of projections with l.u.b. 1 such that

$$h_0(e_i y) \leq n(e_i y) \quad \text{for all } i \in I,$$

then we have

$$h_0(y)=\sum_I e_ih_0(y)=\sum_I e_ih_0(e_iy)\leq\sum_I e_in(e_iy)$$
$$=\sum_I e_in(y)=n(y),$$

which shows that $P(y)$ satisfies (4).

For any non-zero e' in P and y in $D(h_0)$, we can find a non-zero $e''\leq e'$ in P such that $P(e''y)$ is true; that is, $P(y)$ satisfies (5). We shall prove this as follows.

(i) When there exists a non-zero $e''\in P$ such that $e''\leq e'$ and $e''a\geq pe''$ for a positive number p, we put $b=((1-e'')+e''a)^{-1}$. By (7), $n(bx+x_0)-f_0(bx)\geq d$. Since $e''a\geq 0$, we have

$$e''ad\leq e''a(n(bx+x_0)-f_0(bx))=n(e''x+e''ax_0)-f_0(e''x)$$

or

$$h_0(e''y)=e''ad+f_0(e''x)\leqq n(e''(x+ax_0))=n(e''y).$$

Thus $P(e''y)$ is true.

(ii) When there exists a non-zero $e''\in P$ such that $e''\leq e'$ and $e''a\leq -pe''$ for a positive number p, we can show that $P(e''y)$ is true, by the similar method as in (i).

(iii) If both of the cases (i) and (ii) do not hold, then $e'a=0$. Hence $h(e'y)=f_0(e'x)\leq n(e'x)=n(e'y)$.

Therefore, by Lemma 1, $P(y)$ is true; that is, h_0 is n-bounded.

Thus we have $(h_0, D(h_0))\in R$ and $(h_0, D(h_0))\geq(f_0, D(f_0))$. By the maximality of $(f_0, D(f_0))$, we have $D(h_0)=D(f_0)$ and so $x_0\in D(f_0)$, which contradicts the assumption that $x_0\notin D(f_0)$. q.e.d.

4. We state some applications of Theorem 1. Let M be a B^*-algebra with unit 1 and A be a commutative AW^*-algebra. We assume that

(8) *A is the B^*-subalgebra of the center of M and $1\in A$.*

We shall denote by N (or B) the totality of self-adjoint elements in M (or A). If we define as usual that $x\geq 0$ ($x\in N$) if and only if x has non-negative spectra, then N is a semi-ordered vector space (cf. Fukamiya [1]) and the induced ordering in $B\subseteq N$ is coincident with the ordering stated in §1.

According to Nakamura and Turumaru [4], an *expectation* e is a mapping of M satisfying

(9) $e(\alpha x+\beta y)=\alpha e(x)+\beta e(y)$,

(10) $e(x^*)=e(x)^*$,

(11) $x\geq 0$ implies $e(x)\geq 0$,

(12) $e(e(x)y)=e(x)e(y)$,

(13) $e(1)=1$;

and we denote by $E(M, A)$ the totality of expectations on M such that $e(M)=A$. If $e\in E(M, A)$, then $e(ax)=ae(x)$ ($a\in A$). In the case when A is the complex number field C, $E(M, C)$ is the state space of M.

5. We define

(14) $n(x)=\text{g.l.b.}\,(a;\, a\in B,\, x\leq a)$ for x in N.

The g.l.b. is taken in B. Noticing that $x\geq 0$ implies $ex\geq 0$ and $\text{g.l.b.}_i ex_i = e(\text{g.l.b.}_i x_i)$ in B ($e\in P$, $x, x_i\in B$), we can easily verify that

Lemma 2. *(a) $n(x)$ is an n-mapping on N considered as B-module. (b) $\|n(x)\|\leq\|x\|$. (c) $n(0)=0$, $n(1)=-n(-1)=1$.*

We shall call this n-mapping *canonical.*

Lemma 3. *$n(y)=0$ ($y\in N$, $y\geq 0$) implies $y=0$ if and only if*

(15) *for the orthogonal system (e_i; $i\in I$) of projections in A with l.u.b. 1, $e_ix=0$ for all i implies $x=0$ ($x\in M$).*

Proof. The proof of necessity is as follows. Suppose $e_ix=0$ for all $i\in I$, then $e_in(xx^*)=n(e_ixx^*)=0$ for all $i\in I$. Since (15) holds for $x\in A$, we have $n(xx^*)=0$ and so $xx^*=0$. Thus $x=0$.

To prove the sufficiency, suppose $n(y)=0$ ($y\in N$, $y\geq 0$). Let m be $1, 2,\cdots$ and $P_m(y)$ be the proposition that $1/m-y\geq 0$. As (a; $a\in B$, $y\leq a$) forms decreasing directed nets with ordered limit 0, so we can find for any non-zero projection e in B a non-zero projection $e'\leq e$ in B and a in B with $y\leq a$ such that $\|e'a\|\leq 1/m$ or $e'a\leq 1/m$ (cf. Widom [6]). This proves that $P_m(e'y)$ is true.

Let (e_i; $i\in I$) be the family of orthogonal projections in B with l.u.b. 1 such that $P_m(e_iy)$ is true for all $i\in I$ and $1/m-y=z-w$, where z (or w) is the positive (or negative) part of $1/m-y$, then e_iz (or e_iw) is the positive (or negative) part of $e_i(1/m-y)$ and hence $e_iw=0$ for all $i\in I$. From (15), $w=0$ and, hence, $P_m(y)$ is true.

By Lemma 1, we get $P_m(y)$ is true for all m, that is, $0\leq y\leq 1/m$. Thus $y=0$. This completes the proof.

We can easily verify

Lemma 4. *An A-module homomorphism e of M into A is in $E(M, A)$ if and only if e is n-bounded with respect to the canonical n on N.*

6. We shall say that a commutative AW^*-algebra A with (8) is *regularly imbedded* in M if (15) is satisfied. Obviously C is always regularly imbedded. When M itself is an AW^*-algebra and A is an AW^*-subalgebra with (8), A is regularly imbedded.

Now we state

Theorem 2. *In order that $E(M, A)$ is total, that is, $e(xx^*)=0$ for all e in $E(M, A)$ and x in M implies $x=0$, it is necessary and sufficient that A is regularly imbedded in M.*

Proof. If $E(M, A)$ is total, then we can find an AW^*-module H over A on which M acts as a uniformly closed operator algebra (cf. Widom [6]). From this (15) follows immediately.

To prove the sufficiency, we have only to construct $e_y\in E(M, A)$ such that $e_y(y)\neq 0$ for any y in N with $y\geq 0$, $\neq 0$. Let $N_0=By$. Put $e'(ay)=an(y)$ ($a\in B$, n the canonical n-mapping). If $ay=0$, then

$$\|e'(ay)\|^2=\|(e'(ay))(e'(ay))^*\|=\|a^*an(y)n(y)^*\|$$
$$=\|n(a^*ay)n(y)^*\|=0,$$

or $e'(ay)=0$. Thus e' is a uniquely defined B-module homomorphism.

Let $e_1, e_2 \in P$ be $e_1+e_2=1$, $e_1e_2=0$ and $e_1a\geq 0$, $e_2a\leq 0$. From (2) and $-n(-y)\leq n(y)$, we have

$$e_1e'(ay)=e_1an(y)=n(e_1ay)=e_1n(ay),$$
$$e_2e'(ay)=e_2an(y)=-(-e_2a)n(y)=-n(-e_2ay)\leq n(e_2ay)=e_2n(ay).$$

Thus we get $e'(ay)\leq n(ay)$, that is, e' is n-bounded. By Theorem 1, e' is extendible to the whole N preserving n-boundedness, say e. Since $w\in M$ is decomposed uniquely as $w=w_1+iw_2$ ($w_1, w_2\in N$), we can define $e_y(w)=e(w_1)+ie(w_2)$. Then e_y is an A-module homomorphism and n-bounded on N. By Lemma 4, $e_y\in E(M, A)$. By Lemma 3, $e_y(y)=n(y)\neq 0$. q.e.d.

7. A mapping of M with (9)–(12) is called a *quasi-expectation* (cf. [4]). We denote by $QE(M, A)$ the totality of quasi-expectations on M such that $e(M)=A$. We also denote by $H(M, A)$ the totality of A-module homomorphisms of M into A which are continuous in the norm topology.

Theorem 3. *If A is regularly imbedded in M, then $H(M, A)$ is spanned algebraically by $QE(M, A)$.*

Proof. Our proof is a modification of that by Takeda [5] in the case $A=C$.

Let f be a B-module homomorphism of N into B with $\|f(x)\|\leq\|x\|$. To establish our theorem it is sufficient to prove that f is the difference of the two positiveness-preserving B-module homomorphisms.

Put $S=QE(M, A)$. We denote by $m(S)$ the set of all B-valued functions $x(s)$ with $(x(s); s\in S)$ is bounded. $m(S)$ is a B-module with the obvious n-mapping

$$n(x)=\text{l.u.b.}((x(s)x(s)^*)^{1/2}; s\in S).$$

We define a semi-order $x\geq 0$ in $m(S)$ by $x(s)\geq 0$ for all $s\in S$.

As is easily seen, N is embedded in $m(S)$ by the correspondence $x\to x(s)\equiv s(x)\in m(S)$ for x in N. By Theorem 2, M can be considered as acting on an AW^*-module over A. Using this fact, we can show that (α) the induced ordering in N by $m(S)$ is coincident with the original one in N, and that (β) $\|x\|=\|n(x)\|$, modifying the usual proof in the scalar case.

As B is a lattice, we can conclude that

(16) $m(S)$ forms a lattice whose operations are compatible with the B-module operations.

On the other hand, $\|f(x)\|\leq\|x\|$ implies $f(x)\leq n(x)$. In fact, let a be an arbitrary invertible element in B such that $n(x)\leq a$, then $\|f(a^{-1}x)\|\leq\|a^{-1}x\|=\|n(a^{-1}x)\|=\|a^{-1}n(x)\|\leq 1$. From this we have

$f(a^{-1}x)=a^{-1}f(x)\leq 1$ or $f(x)\leq a\downarrow n(x)$. Thus f is n-bounded on N and, by Theorem 1, it can be extended to whole $m(S)$ preserving n-boundedness. We denote it again by f. As $x\geq y\geq 0$ in $m(S)$ implies $n(x)\geq n(y)\geq f(y)$, we can define for $x\geq 0$

$$e_1(y)=\text{l.u.b.}(f(y);\ x\geq y\geq 0,\ y\in m(S)).$$

By (16) we can apply the known argument in the theory of vector lattices to prove that e_1 is extendible to the whole $m(S)$ naturally. Let

$$e_2(x)=e_1(x)-f(x) \qquad (x\in m(S)).$$

It is not so hard to see that e_1 and e_2 are B-module homomorphisms of $m(S)$ and hence of N. By definition, it is also easy to see that $e_i(x)\geq 0$ $(i=1,2)$ for $x\geq 0$ in $m(S)$. Thus, the restriction of e_i on N gives the desired decomposition $f=e_1-e_2$. q.e.d.

Remark. Let M be a B^*-algebra with or without the unit and A be an AW^*-algebra being commutative but not necessarily contained in M.

Theorems 2 and 3 are extended to the case when A satisfies (15) and the following condition instead of (8):

(8′) *M is an associative algebra over A with $a(xy)=(ax)y=x(ay)$ and $||ax||\leq||a||\ ||x||$ $(a\in A,\ x, y\in M)$.*

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