107. A Remark on My Paper "A Boundary Value Problem of Partial Differential Equations of Parabolic Type" in Duke Mathematical Journal

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§ 1. **Introduction.** Recently Dr. T. Shirota kindly called the author's attention to the following fact; — in the author's paper [1] published in Duke Mathematical Journal, 24, the continuity of $p_z(t, x; s, y)$ — and accordingly that of the fundamental solution $u(t, x; s, y)$ — in $y \in \boldsymbol{B}$ is not obvious in the case where $\alpha(t, \xi)$ takes the value zero for some $\langle t, \xi \rangle$ and is not identically zero. The same situation occurs in the author's another paper [2]. In the present note, instead of completing the proof of the continuity of the fundamental solution, we shall slightly modify the argument in [1].

The argument in the present note may be adapted to [2]. By the way, we state the following correction to the paper [2]; — $[1-p_z(s, y; t, x)]$ *in the numerator of the right-hand side of* (3.24) *in* [2, p. 63] *should be replaced by* $p_z(s, y; t, x)$.

§ 2. **Construction of the fundamental solution.** We shall use notations stated in [1] without repeating definitions of them. We first notice that, if $\alpha(t, \xi)$ identically equals zero or is bounded away from zero, $p_z(t, x; s, y)$ has desired regularity and accordingly $u(t, x; s, y)$ does.

For each $n \geq 1$, let $\chi_n(\lambda)$ be a monotone increasing function of class C^3 in $\lambda \in [0, 1]$ such that

$$(1) \qquad \begin{aligned} \chi_n(\lambda) &= 1/(n+1) \qquad \text{for } \lambda \leq 1/(n+2) \\ &= \lambda \qquad\qquad\quad \text{for } \lambda \geq 1/n. \end{aligned}$$

We define $\alpha_n(t, \xi)$ and $\beta_n(t, \xi)$ on $[s_0, t_0] \times \boldsymbol{B}$ for $n = 0, 1, 2, \cdots$ as follows:

$$(2) \qquad \begin{cases} \alpha_0(t, \xi) = 0, \quad \alpha_n(t, \xi) = \chi_n(\alpha(t, \xi)) \quad (n \geq 1) \\ \text{and} \quad \beta_n(t, \xi) = 1 - \alpha_n(t, \xi) \quad (n = 0, 1, 2, \cdots) \end{cases}$$

where $\alpha(t, \xi)$ is the function stated in the given boundary condition (B_φ^t) in [1]. Then, for each $n \geq 0$, we may apply the argument in [1] to the parabolic equation $Lf + h = 0$ associated with boundary condition:

$$(B_{n,\varphi}^t) \qquad \alpha_n(t, \xi) f(t, \xi) + \beta_n(t, \xi) [\partial f(t, \xi) / \partial \boldsymbol{n}_{t,\xi}] = \varphi(t, \xi),$$

and obtain the fundamental solution $u_n(t, x; s, y)$ with all properties stated in [1] where (B_φ^t) is replaced by $(B_{n,\varphi}^t)$.

Let $f(x)$ be an arbitrary continuous and non-negative function on $\overline{\boldsymbol{D}}$ and put

$$(3)\qquad f_n(t,x)=\int_D u_n(t,x;s,y)f(y)dy \qquad (n\geq 0)$$

and

$$(4)\qquad \varphi_{n\nu}(t,\xi)=\alpha_n(t,\xi)f_\nu(t,\xi)+\beta_n(t,\xi)\frac{\partial f_\nu(t,\xi)}{\partial \boldsymbol{n}_{t,\xi}} \qquad (\nu\geq n\geq 0).$$

Then $\beta_\nu\varphi_{n\nu}=(\alpha_n\beta_\nu-\alpha_\nu\beta_n)f_\nu=(\alpha_n-\alpha_\nu)f_\nu$, and hence

$$(5)\qquad \varphi_{n\nu}(t,\xi)=\begin{cases}[\alpha_n(t,\xi)-\alpha_\nu(t,\xi)][1-\alpha_\nu(t,\xi)]^{-1}f_\nu(t,\xi) & \text{if } \alpha(t,\xi)\leq 1/n\\ 0 & \text{if } \alpha(t,\xi)>1/n\end{cases}$$

and

$$(6)\qquad \varphi_{0\nu}(t,\xi)=\begin{cases}-\alpha_\nu(t,\xi)[1-\alpha_\nu(t,\xi)]^{-1}f_\nu(t,\xi) & \text{if } \alpha(t,\xi)\neq 1\\ \partial f_\nu(t,\xi)/\partial\boldsymbol{n}_{t,\xi} & \text{if } \alpha(t,\xi)=1\end{cases}$$

for $\nu\geq n\geq 1$. Furthermore, since $f_\nu(t,x)-f_n(t,x)$ satisfies the equation $L[f_\nu-f_n]=0$ on $(s,t_0)\times\overline{\boldsymbol{D}}$, initial condition: $\lim_{t\downarrow s}[f_\nu-f_n]=0$ boundedly on $\boldsymbol{D}$, and boundary condition $(\mathrm{B}^t_{n,\phi})$ with $\phi=\varphi_{n\nu}$, we have (see part iii of Theorem in [1])

$$(7)\qquad \begin{aligned} f_\nu(t,x)-f_n(t,x)=\int_s^t d\tau\int_B\{&u_n(t,x;\tau,\xi)[1+\phi(\tau,\xi)]\\ &-\partial u_n(t,x;\tau,\xi)/\partial\boldsymbol{n}_{\tau,\xi}\}\varphi_{n\nu}(\tau,\xi)d'_\tau\xi.\end{aligned}$$

Since $u_n(t,x;s,y)$ is non-negative ((1.5) in [1]) and satisfies the boundary condition of the form (4.12) in [1] as a function of $\langle s,y\rangle$, the value of the function in { } in the right-hand side of (7) is always non-negative, while $\varphi_{n\nu}(\tau,\xi)\geq 0\geq\varphi_{0\nu}(\tau,\xi)$ for $\nu\geq n\geq 1$ by virtue of (5) and (6). Hence we have

$$f_n(t,x)\leq f_\nu(t,x)\leq f_0(t,x) \qquad \text{for } \nu\geq n\geq 1,$$

and hence

$$u_n(t,x;s,y)\leq u_\nu(t,x;s,y)\leq u_0(t,x;s,y) \qquad \text{for } \nu\geq n\geq 1$$

since $f(x)$ is arbitrary in (3). Therefore

$$(8)\qquad u(t,x;s,y)=\lim_{n\to\infty}u_n(t,x;s,y)$$

exists and does not exceed $u_0(t,x;s,y)$.

It follows from (3), (5) and (7) that

$$(9)\qquad \begin{aligned} u(t,x;s,y)-u_n(t,x;s,y)=\int_s^t d\tau\int_B\{&u_n(t,x;\tau,\xi)[1+\Psi(\tau,\xi)]\\ &-\partial u_n(t,x;\tau,\xi)/\partial\boldsymbol{n}_{\tau,\xi}\}\Phi_{n\nu}(\tau,\xi;s,y)d'_\tau\xi\end{aligned}$$

where

$$(10)\qquad \Phi_{n\nu}(t,\xi;s,y)=\begin{cases}[\alpha_n(t,\xi)-\alpha_\nu(t,\xi)][1-\alpha_\nu(t,\xi)]^{-1}u_\nu(t,\xi;s,y) & \text{if } \alpha(t,\xi)\leq 1/n\\ 0 & \text{if } \alpha(t,\xi)>1/n\end{cases}$$

for $\nu\geq n\geq 1$. Letting $\nu\to\infty$, we obtain

$$u(t, x; s, y) - u_n(t, x; s, y)$$

$$(11) \qquad = \int_s^t d\tau \int_B \{u_n(t, x; \tau, \xi)[1 + \Psi(\tau, \xi)] - \partial u_n(t, x; \tau, \xi)/\partial \boldsymbol{n}_{\tau,\xi}\} \Phi_n(\tau, \xi; s, y) d'_\tau \xi$$

where

$$(12) \qquad \Phi_n(t, \xi; s, y) = \begin{cases} [\alpha_n(t, \xi) - \alpha(t, \xi)][1 - \alpha(t, \xi)]^{-1} u(t, \xi; s, y) & \text{if } \alpha(t, \xi) \leq 1/n, \\ 0 & \text{if } \alpha(t, \xi) > 1/n \end{cases}$$

for $n \geq 1$. Since $u_n(t, x; s, y)$ has all properties stated in Theorem in [1, § 1] where (B^t_φ) is replaced by $(B^t_{n,\varphi})$, it follows from (11) and (12) that $u(t, x; s, y)$ satisfies (1.1–7) and (3.13) in [1] — we shall prove only (1.2); all other properties may be proved more easily.

Part ii) of Theorem in [1] and (11) imply that $u(t, x; s, y) - u_n(t, x; s, y)$ satisfies the boundary condition (B^t_{n,φ_n}) with $\varphi_n(t, \xi) = \Phi_n(t, \xi; s, y)$ for any fixed $\langle s, y \rangle$. Hence, at any point $\langle t, \xi \rangle$ where $\alpha(t, \xi) > 0$, $u(t, x; s, y)$ satisfies (1.2) in [1] as well as $u_n(t, x; s, y)$ since $\alpha_n(t, \xi) = \alpha(t, \xi)$, $\beta_n(t, \xi) = \beta(t, \xi)$ and $\varphi_n(t, \xi) = 0$ for sufficiently large n. At any point $\langle t, \xi \rangle$ where $\alpha(t, \xi) = 0$, we have $\alpha_n(t, \xi) = (n+1)^{-1}$ (from (1) and (2)) and accordingly

$$(n+1)^{-1} u(t, \xi; s, y) + \{1 - (n+1)^{-1}\} \frac{\partial u(t, \xi; s, y)}{\partial \boldsymbol{n}_{t,\xi}} = \Phi_n(t, \xi; s, y) = (n+1)^{-1} u(t, \xi; s, y).$$

Hence we get $\partial u(t, \xi; s, y)/\partial \boldsymbol{n}_{t,\xi} = 0$. Thus we obtain (1.2) in [1].

Similarly we may construct a function $u^*(t, x; s, y)$ satisfying (1.1*, 2*, 4*) and (3.13*) in [1] and, repeating the argument in [1, § 4], we may show that $u(t, x; s, y)$ has all required properties.

References

[1] S. Itô: A boundary value problem of partial differential equations of parabolic type, Duke Math. J., **24**, 299–312 (1957).

[2] S. Itô: Fundamental solutions of parabolic differential equations and boundary value problems, Jap. J. Math., **27**, 55–102 (1957).