# 138. On a Generalization of the Concept of Functions. II

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In our previous paper [1], we have introduced the notion of *hyperfunctions* on $C^{\omega}$-manifolds by means of boundary values of analytic functions as a generalization of the concept of functions, and sketched the theory thereof in case of dimension 1 (the theory of *hyperfunctions of a single variable*). The purpose of the present and subsequent papers is to give the outline of the theory in case of dimensions $>1$ (the theory of *hyperfunctions of several variables*).[1]

§1. *Distributions of a sheaf.* Let $X$ be a topological space. We denote with $\mathfrak{L}(X)$ the totality of open sets of $X$. Let $\mathfrak{F}$ be a sheaf of modules over $X$. For any $D\in\mathfrak{L}(X)$ and $n=0,1,2,\cdots$, we denote as usual the $n$-cohomology group of $D$ with coefficients in $\mathfrak{F}$ with $H^n(D,\mathfrak{F})$. $H^0(D,\mathfrak{F})$ is the section-module of $\mathfrak{F}$ over $D$.

Let $S$ be a closed subset of $X$. For any $D\in\mathfrak{L}(X)$ and $n=0,1,2,\cdots$, we define $G^n(S,D,\mathfrak{F})$ as follows: $G^0(S,D,\mathfrak{F})$ and $G^1(S,D,\mathfrak{F})$ are to mean the kernel and cokernel of the natural homomorphism $H^0(D,\mathfrak{F})\to H^0(D-S,\mathfrak{F})$ respectively, and for $n\geq 2$ we put $G^n(S,D,\mathfrak{F})=H^{n-1}(D-S,\mathfrak{F})$.

For $D\supset D'(D',D\in\mathfrak{L}(X))$ we have the natural homomorphism $\rho^n_{D'D}$: $G^n(S,D,\mathfrak{F})\to G^n(S,D',\mathfrak{F})$. For each $n$, $(\{G^n(S,D,\mathfrak{F})\}_{D\in\mathfrak{L}(X)}, \{\rho^n_{D'D}\}_{D',D\in\mathfrak{L}(X)})$ constitutes a pre-sheaf over $X$. We shall denote with $\mathrm{Dist}^n(S,X,\mathfrak{F})$ the sheaf over $X$ determined by this pre-sheaf. $\mathrm{Dist}^n(S,X,\mathfrak{F})$ has the stalk 0 at any point on $X-S$, and if $X'\in\mathfrak{L}(X)$, $X'\supset S$, the natural homomorphism $\mathrm{Dist}^n(S,X,\mathfrak{F})\to\mathrm{Dist}^n(S,X',\mathfrak{F})$ is clearly bijective. In identifying these $\mathrm{Dist}^n(S,X',\mathfrak{F})$, we shall denote the sheaf over $S$ thus determined by $\mathrm{Dist}^n(S,\mathfrak{F})$.

*Definition* 1. We call each element of $H^0(S,\mathrm{Dist}^n(S,\mathfrak{F}))=H^0(X,\mathrm{Dist}^n(S,X,\mathfrak{F}))$ an *$\mathfrak{F}$-distribution of degree n* over $S$.

It is clear that we have the natural homomorphism:

$$G^n(S,X,\mathfrak{F})\to H^0(S,\mathrm{Dist}^n(S,\mathfrak{F})) \tag{1}$$

which is bijective for $n=0$.

*Example* 1. For $S=X$, we have $\mathrm{Dist}^0(S,\mathfrak{F})=\mathfrak{F}$, $H^0(S,\mathrm{Dist}^0(S,\mathfrak{F}))=H^0(X,\mathfrak{F})$, while $\mathrm{Dist}^n(S,\mathfrak{F})=0$ for $n\geq 1$.

Now let $\{X',\mathfrak{F}',S'\}$ be another triple consisting of a topological space $X'$, a sheaf of modules $\mathfrak{F}'$ over $X'$, and a closed set $S'$ of $X'$.

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1) We have explained our theory, including the case of several variables, in [2] in Japanese. An English account will soon appear in J. Fac. Sci. Univ. Tokyo.

Let $\sigma$ be a continuous mapping from $(X', X'-S')$ into $(X, X-S)$ (i.e. a continuous mapping from $X'$ into $X$ such that $\sigma(X'-S')\subset X-S$), and $\theta$ be a homomorphism from $\mathfrak{F}$ into $\mathfrak{F}'$ compatible with $\sigma$. Then $\theta$ induces in a natural manner the homomorphism

$$(2)\qquad \theta^*:\ \mathrm{Dist}^n(S, \mathfrak{F})\to \mathrm{Dist}^n(S', \mathfrak{F}')$$

compatible with $\sigma$. In particular, suppose $X'\in\mathfrak{L}(X)$, $X'\frown S\subset S'$, $\sigma=$injection $X'\to X$, and $\mathfrak{F}'=\mathfrak{F}\,|\,X'=$restriction of $\mathfrak{F}$ onto $X'$. Then the natural homomorphism $\mathrm{Dist}^n(S, \mathfrak{F})\to \mathrm{Dist}^n(S', \mathfrak{F}')$ is induced, and hence also the homomorphism $H^0(S, \mathrm{Dist}^n(S, \mathfrak{F}))\to H^0(S', \mathrm{Dist}^n(S', \mathfrak{F}'))$. We shall denote with $g\,|\,S'$ the image of $g\in H^0(S, \mathrm{Dist}^n(S, \mathfrak{F}))$ by this homomorphism. Setting $S''=X'\frown S$, we say that $g''=g\,|\,S''\in H^0(S', \mathrm{Dist}^n(S'', \mathfrak{F}'))$ is the *restriction* of $g$ onto $S''$ (which is an *open* subset of $S$), and $g'=g''\,|S'(=g\,|\,S')\in H^0(S', \mathrm{Dist}^n(S', \mathfrak{F}'))$ is the *dilution* of $g''$ onto $S'$ (which contains $S''$ as a *closed* subset).

*Representation by Čech cohomology.* In order to obtain a more concrete representation of $\mathfrak{F}$-distributions over $S$ we shall invoke to Čech's cohomology theory. Let $(\mathfrak{U}, \mathfrak{U}')$ be an open covering of $(X, X-S)$, i.e. let $\mathfrak{U}=\{U_\alpha;\ \alpha\in N\}$ be an open covering of $X$, and $\mathfrak{U}'=\{U_\alpha;\ \alpha\in N'\}$, $N'\subset N$, a subset of $\mathfrak{U}$, which constitutes an open covering of $X-S$. For such $(\mathfrak{U}, \mathfrak{U}')$, we can define the cohomology groups $H^n(\mathfrak{U}, \mathfrak{F})$ and $H^n(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F})$ as usual in the following way. An $n$-cochain $\varphi\in C^n(\mathfrak{U}, \mathfrak{F})$ will be defined as a vector

$$(3)\qquad \varphi=(\varphi_{\alpha_0\cdots\alpha_n}),$$

with the components $\varphi_{\alpha_0\cdots\alpha_n}\in H(U_{\alpha_0}\frown\cdots\frown U_{\alpha_n}; \mathfrak{F})$ where we shall assume, without loss of generality, that $\varphi_{\alpha_0\cdots\alpha_n}$ are alternating for the permutation of suffices. Define the coboundary operator $\delta=\delta_N$ in a well-known manner. Then $C(\mathfrak{U}, \mathfrak{F})=\sum_{n\ge 0}C^n(\mathfrak{U}, \mathfrak{F})$ constitutes a $DG$-module,[2] and $H^n(\mathfrak{U}, \mathfrak{F})$ is defined by

$$(4)\qquad H^n(\mathfrak{U}, \mathfrak{F})=H^n(C(\mathfrak{U}, \mathfrak{F})).$$

We obtain $H^n(X, \mathfrak{F})$ as the inductive limit of $H^n(\mathfrak{U}, \mathfrak{F})$ by refining the covering $\mathfrak{U}$. The relative cochain group $C^n=C^n(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F})$ consists of such $\varphi\in C^n(\mathfrak{U}, \mathfrak{F})$ whose components $\varphi_{\alpha_0\cdots\alpha_n}$ are all 0 for $\alpha_0, \cdots, \alpha_n\in N'$. $\sum_{n\ge 0}C^n=C(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F})$ is a $DG$-submodule of $C(\mathfrak{U}, \mathfrak{F})$, and $H^n(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F})$ is defined by

$$(5)\qquad H^n(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F})=H^n(C(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F})).$$

Then we obtain $H^n(X \bmod (X-S), \mathfrak{F})$ as the inductive limit of $H^n(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F})$ by refining $(\mathfrak{U}, \mathfrak{U}')$, and we have the excision theorem

$$(6)\qquad H^n(X \bmod (X-S), \mathfrak{F})\simeq H^n(X' \bmod (X'-S), \mathfrak{F})\ \text{ if }\ S\subset X'\in\mathfrak{L}(X).$$

Now introduce a filtration-structure into $C=C(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F})$ as follows. Let ${}^pC^n={}^pC^n(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F})$ be the submodule of $C^n$ consisting of $\varphi=(\varphi_{\alpha_0\cdots\alpha_n})$ such that $\varphi_{\alpha_0\cdots\alpha_n}=0$ if at least $n+1-p$ elements of $\{\alpha_0, \cdots, \alpha_n\}$

2) On the definition of $DG$-modules and other related notions, see [3].

are in $N'$, then $C$ constitutes an *FDG*-module having $p$ and $n$ as the filtration degree and the total degree respectively. We shall define, as usual,

(7) ${}^pZ_r^n={}^pZ_r^n(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F})=$inverse image of ${}^{p+r}C^{n+1}$ in the homomorphism

$$\delta:\ {}^pC^n \to {}^pC^{n+1},$$

(8) $${}^pE_r^n={}^pE_r^n(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F})={}^pZ_r^n \bmod({}^{p+1}Z_{r-1}^n, \delta^{p-r+1}Z_{r-1}^{n-1}).$$

For $r\geq 2$, we obtain ${}^pE_r^n(X \bmod(X-S), \mathfrak{F})$ as the inductive limit of ${}^pE_r^n(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F})$ by refining $(\mathfrak{U}, \mathfrak{U}')$.[3] Among these, ${}^pE_2^n(X \bmod(X-S), \mathfrak{F})$ is of particular importance, as we have a natural homomorphism

(9) $${}^pE_2^n(X \bmod(X-S), \mathfrak{F}) \to H^p(S, \mathrm{Dist}^n(S, \mathfrak{F})).$$

*Proposition* 1. *If $X$ and $X-S$ are both paracompact $T_2$-spaces, then the homomorphism* (9) *is bijective. In particular, every $\mathfrak{F}$-distribution $g$ of degree $n$ over $S$ corresponds to an element of ${}^0E_2^n$ $(X \bmod (X-S), \mathfrak{F})$ in a 1–1 manner.*

Thus, under the assumptions of Proposition 1, each $g \in H^0(S, \mathrm{Dist}^n(S, \mathfrak{F}))$ is determined by some $(\mathfrak{U}, \mathfrak{U}')$ covering $(X, X-S')$ and some $\varphi \in {}^0Z_2^n(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F})$. We call this $\varphi$ a defining function of $g$, and denote

(10) $$g=[\varphi]=[\varphi, \mathfrak{U}, \mathfrak{U}'].$$

§ 2. *Analytic distributions.* Let $X=X^m$ be an analytic manifold of complex dimension $m$, and $\mathfrak{F}$ be a locally free analytic sheaf over $X$, i.e. a sheaf consisting of the analytic local sections of some complex analytic vector bundle $\boldsymbol{B}$ over $X$. Let $S$ be a closed subset of $X$.

*Definition* 2. *An analytic distribution $g$* of degree $n$ (in short: an $n$-distribution) of type $\boldsymbol{B}$ over $S$ is an $\mathfrak{F}$-distribution of degree $n$ over $S$.

If $\boldsymbol{B}=X\times\boldsymbol{C}$, ($\boldsymbol{C}$=complex number field), the qualifying phrase 'of type $\boldsymbol{B}$' will be omitted, and if $\boldsymbol{B}$ is a vector bundle of differential forms, of tensors, or of differential operators etc. (of some given type, respectively), then the analytic distributions of the corresponding type $\boldsymbol{B}$ will be called the analytic distributions of differential forms, of tensors, of differential operators, etc.

*Proposition* 2. (i) *Analytic distributions of degree $n>m$ other than* 0 *of any type $\boldsymbol{B}$ do not exist. Namely, we have* $\mathrm{Dist}^n(S, \mathfrak{F})=0$, $(n>m)$, *for germs of analytic $n$-distributions over $S$.*

(ii) $\mathrm{Dist}^m(S, \mathfrak{F})$ *is a complete sheaf*[4] *for any type $\boldsymbol{B}$ and closed set $S$.*

3) Moreover, we can construct a spectral sequence connecting ${}^pE_2^n(X \bmod(X-S), \mathfrak{F})$ to $H^n(X \bmod(X-S), \mathfrak{F})$.

4) A sheaf $\mathfrak{F}$ is called *complete* or *hyperfine* if the natural homomorphism $H^0(D, \mathfrak{F}) \to H^0(D', \mathfrak{F})$ is surjective for every $D\supset D'$, $(D, D' \in \mathfrak{L}(X))$. A complete sheaf is always a fine sheaf, but the converse is not true. For instance, all the following sheaves are fine but not complete: the sheaf of germs of $C^\infty$-functions over a $C^\infty$-manifold; the sheaf of germs of Schwartz distributions over a $C^\infty$-manifold; the sheaf of germs of continuous functions over a $C^0$-manifold; the sheaf of germs of locally integrable functions over a $C^1$-manifold.

*Integration.* Let us consider the particular case where $S \subset X^m$ is a compact set such that analytic distributions of degree $<m$ over $S$ other than 0 do not exist. Then we can define the *definite integral* for an analytic $m$-distribution of *m-differential forms.* Replacing $X$ by a suitable $X'$ such that $S \subset X' \in \mathfrak{L}(X)$ if necessary, we may assume that $X$ is perfectly separable (and so $X$ is paracompact) from the beginning. We may therefore put $g=[\varphi, \mathfrak{U}, \mathfrak{U}']$, $\varphi \in {}^0Z_2^m(\mathfrak{U} \bmod \mathfrak{U}', \mathfrak{F})$, where we can further assume that $\mathfrak{U}$ is a locally finite covering and $\mathfrak{U}-\mathfrak{U}'$ is finite. Now, since $X=X^m$ is an orientable completely separable differentiable manifold of real dimension $2m$, we can find differentiable $2m$-chains $V_\alpha$, $(\alpha \in N)$, such that (i) $|V_\alpha| \subset U_\alpha$ for every $\alpha \in N$, (ii) $|V_\alpha|$ is compact if $\alpha \in N'$, (iii) $\sum_{\alpha \in N} V_\alpha =$ the fundamental cycle of $X$. From these $2m$-chains, we can further find differentiable $(2m-n)$-chains $V_{\alpha_0 \cdots \alpha_n}$ such that $|V_{\alpha_0 \cdots \alpha_n}| \subset U_{\alpha_0} \frown \cdots \frown U_{\alpha_n}$ with $n=1, 2, \cdots$ satisfying the following relations:

i) $\partial V_{\alpha_0} = \sum_{\alpha_1} V_{\alpha_0 \alpha_1}$, $\partial V_{\alpha_0 \alpha_1} = \sum_{\alpha_2} V_{\alpha_0 \alpha_1 \alpha_2}, \cdots$,

ii) $V_{\alpha_0 \cdots \alpha_n}$ is alternating for the permutations of suffices $\alpha_0, \cdots, \alpha_n$.

*Definition* 3. We define the definite integral of $g$ by

$$\underbrace{\int \cdots \int_S}_{m\text{-uple}} g = \sum_{(\alpha_0 \cdots \alpha_m)}' \underbrace{\int \cdots \int}_{m\text{-uple}}{}_{V_{\alpha_0 \cdots \alpha_m}} \varphi_{\alpha_0 \cdots \alpha_m}. \tag{11}$$ [5]

That the value of the integral does not depend on the choice of $(\varphi, \mathfrak{U}, \mathfrak{U}')$ and $\{V_\alpha\}$ follows from the integration theorem of Cauchy-Poincaré.

The notion of integration defined by (9) may be generalized in the following case.

Let $\sigma$ be an open analytic mapping from $(X, X-S)$ to $(X', X'-S')$, and assume that for each $x \in X'$ the inverse image $\sigma^{-1}(x)$ is a sub-variety of $X$ without singularity (of complex dimension $n=m-m'>0$, $m$ and $m'$ denoting the complex dimensions of $X$ and $X'$).

Let $\boldsymbol{B}'$ be an analytic vector bundle over $X'$, $\sigma^{-1}(\boldsymbol{B}')$ be the inverse image of $\boldsymbol{B}'$ (i.e. the induced bundle from $B'$ by $\sigma$). On the other hand, denote with $\boldsymbol{T}_x$ the tangential tensor bundle of rank $n$ over $\sigma^{-1}(x)$ for each $x \in X'$. Then $\boldsymbol{T} = \bigcup_{x \in X'} \boldsymbol{T}_x$ constitutes a sub-bundle of the tangential tensor bundle of rank $n$ over $X$. Accordingly, an analytic vector bundle $\boldsymbol{B}$ over $X$ is defined by $\boldsymbol{B} = \mathrm{Hom}(T, \sigma^{-1}(\boldsymbol{B}'))$.

Let moreover, $S \frown \sigma^{-1}(x)$ be compact for every $x \in X'$, and satisfy a suitable additional condition. Then integrating in a suitably generalized sense any analytic $l$-distribution $g$ of type $\boldsymbol{B}$ over $S$, $l$ being an integer $\geq n$, we have

5) The summation symbol of the right-hand side signifies the sum running over all distinct *oriented* $m$-simplexes $(\alpha_0, \cdots, \alpha_n)$ of $N$ such that at least $n-1$ elements of $\{\alpha_0, \cdots, \alpha_n\}$ are in $N'$. Note that the number of non-vanishing terms is finite.

$$(12) \qquad \underbrace{\int\cdots\int}_{n\text{-uple}}{}_{S\frown\sigma^{-1}(x)} g=g'(x)$$

where the value $g'$ of the integral is an analytic $(l-n)$-distribution of type $\boldsymbol{B}'$ over $S'$.

If another open analytic mapping $\sigma':(X', X'-S')\to(X'', X''-S'')$ and the corresponding integration: $g'\to g''$ are given, then we get the integration: $g\to g''$ corresponding to the mapping $\sigma'\sigma:(X, X-S)\to(X'', X''-S'')$ (the Fubini theorem).

*Hyperfunctions.* Now take $X\in\mathfrak{L}(\boldsymbol{C}^m)$ and $S=X\frown R^m$. Then we have

*Proposition* 3. *Analytic distributions of degree* $n\neq m$ *of any type* $\boldsymbol{B}$ *other than* 0 *over* $S$ *do not exist.*

Therefore we need only consider $m$-distributions on such $S\in\mathfrak{L}(R^m)$. We call these $m$-distributions on $S$ the *hyperfunctions of* $m$ *variables.* In utilizing the fact that any $m$-dimensional real analytic manifold $M^m$ is locally analytically isomorphic with some $S\in\mathfrak{L}(R^m)$, we can extend this notion of hyperfunctions of $m$ variables to the case where the underlying manifold is $M^m$ instead of $S$. We shall develop the theory of these hyperfunctions in a forthcoming note.

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## References

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