155. Remarks on Some Riemann Surfaces

By Kikuji MATSUMOTO

Mathematical Institute, Nagoya University

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1. In the theory of meromorphic functions, it is important to investigate the properties of covering surfaces generated by their inverse functions. For this purpose, the study of properties of a non-compact region of a Riemann surface is useful.

Recently Kuramochi [4] gave interesting results for non-compact regions of some Riemann surfaces and these results were extended by Constantinescu and Cornea [1] and himself [5]. On the other hand, the method given by Heins [2] is also expected to contribute for the same purpose. So, in this note, we shall investigate some properties of covering surfaces using Kuramochi's results and Heins' method. Here we shall omit the details which will appear elsewhere.

2. Let R_1 and R_2 be two Riemann surfaces which do not belong to O_G, and let f be a conformal mapping of R_1 into R_2. We denote by $\mathfrak{G}_{R_1}$ and $\mathfrak{G}_{R_2}$ Green functions of R_1 and R_2 respectively. Then, holds the equality

$$\mathfrak{G}_{R_2}(f(p); q) = \sum_{f(r)=q} n(r)\mathfrak{G}_{R_1}(p; r)+u_q(p),$$

where $n(r)$ is the multiplicity of f at $r\in R_1$, and $u_q(p)$ is the greatest harmonic minorant of $\mathfrak{G}_{R_2}(f(p); q)$ on R_1.

Generally, a positive harmonic function is representable uniquely by the sum of a positive quasi-bounded harmonic function and a positive singular harmonic function (Parreau [7]). Heins [2] proved that $u_q(p)$ is quasi-bounded except for a set of q of capacity zero and that the quasi-bounded component of $u_q(p)$ is either positive on $R_1\times R_2$ or constantly zero.

According to Heins [2], we say that f is of type-Bl if the second alternative occurs for f.

Now, let R_1 and R_2 be arbitrary Riemann surfaces, and let f be a conformal mapping of R_1 into R_2. We shall say that f is of type-Bl at $q\in R_2$ provided that there exists a simply-connected Jordan region Ω satisfying: (1) $q\in \Omega\subset R_2$, (2) $f^{-1}(\Omega)\neq\phi$ and (3) for each component Δ of $f^{-1}(\Omega)$, the restriction f_Δ of f to Δ is of type-Bl considering f_Δ as to be a conformal mapping of Δ into Ω. We shall say that f is locally of type-Bl if f is of type-Bl at each point of R_2.

For simplicity, we shall call a non-compact or compact domain on a Riemann surface R a subregion on R when its relative boundary with respect to R consists of at most an enumerable number of analytic curves being compact or non-compact and clustering nowhere in R. Then, we get the following:

Theorem 1. *Let R_1 and R_2 be arbitrary Riemann surfaces, and let f be a conformal mapping of R_1 into R_2. Then, f is locally of type-Bl if and only if, for any compact subregion Ω on R_2 (we suppose that Ω has at least one exterior point when R_2 is compact), each component of $f^{-1}(\Omega)$ belongs to SO_{HB}.*

3. According to Constantinescu and Cornea [1], we denote by $O_{HB_n}(O_{HD_n})$ $(1\leqq n\leqq\infty)$ the class of Riemann surfaces, the ideal boundary of which consists of at most n HB- (maximal HD-) indivisible sets in their sense. These classes are the same ones considered by Kuramochi [5]. In fact, as Constantinescu and Cornea proved, $O_{HB_n}(O_{HD_n})$ $(1\leqq n<\infty)$ coincides with the class of Riemann surfaces on which there exist at most n number of linearly independent bounded (Dirichlet-bounded) harmonic functions.

Heins [3] introduced a class O_L of Riemann surfaces, on which there exists no non-constant single-valued Lindelöfian meromorphic function. Here, we say a conformal mapping of a Riemann surface R_1 into another Riemann surface R_2 is Lindelöfian if

$$\sum_{f(r)=q} n(r)\mathfrak{G}_{R_1}(p;r)<+\infty$$

for $f(p)\neq q$. It was proved by Heins that the inclusion-relation

$$O_{HB_1}=O_{HB}\subsetneqq O_L\subsetneqq O_{AB}$$

holds and that, for the class of Riemann surfaces with finite genus,

$$O_G=O_{HB}=O_L$$

holds.

Theorem 2 (Kuramochi). *If a Riemann surface R belongs to $O_{HB_n}-O_G$ $(1\leqq n\leqq\infty)$ and the subregion G on R does not belong to SO_{HB}, then G belongs to O_L.*

Kuroda [6] introduced a class O_{AB}^0 of Riemann surfaces whose all subregions belong to SO_{AB} and proved that each Riemann surface belonging to O_{AB}^0 has Iversen property, and the inclusion-relation

$$O_{HB}\subsetneqq O_{AB}^0\subsetneqq O_{AB}$$

holds and that, for the class of Riemann surfaces with finite genus,

$$O_G=O_{HB}\subsetneqq O_{AB}^0\subset O_{AB}.$$

Noticing Theorem 2, we can verify the following:

$$O_L\not\gtreqless O_{AB}^0\quad\text{and}\quad O_L\not\gtreqless O_{HD}.$$

Theorem 3 (Kuramochi). *If a Riemann surface R belongs to $O_{HD_n}-O_G$ $(1\leqq n\leqq\infty)$ and the subregion G on R does not belong to SO_{HD}, then G belongs to O_{AD}.*

4. Here we shall state some results which are deduced from Theorems 1 and 2.

Theorem 4. *If a Riemann surface R belongs to* O_{HB_n} $(1 \leqq n \leqq \infty)$, *then any non-constant single-valued meromorphic function f on R is locally of type-Bl.*

Corollary. *Let R be Riemann surface belonging to* O_{HB_n} $(1 \leqq n \leqq \infty)$, *and let* Φ *be the covering surface of the w-plane generated by a non-constant single-valued meromorphic function f on R. Then every connected piece* Φ_δ *of* Φ *on any disc* δ *on the w-plane covers the same times each point of* δ *except at most an* F_σ*-set of capacity zero.*

Theorem 5. *Let R be a Riemann surface belonging to* O_{HB_n} $(1 \leqq n \leqq \infty)$ *and let G be a subregion on R not belonging to* SO_{HB}. *Then the cluster set of any non-constant single-valued meromorphic function f on G at the ideal boundary of G is total, and the range of values of f contains all values of the w-plane except for at most an* F_σ*-set of capacity zero.*

5. Here we concern with the subsurface on Riemann surfaces of the class O_{HD_n}.

Theorem 6. *Let f be a non-constant single-valued meromorphic function on a Riemann surface R. If there exist a point* w_0 *and the sequence of Jordan regions* Ω_i *of the w-plane such that* $\Omega_i \supset \overline{\Omega}_{i+1}$, $\bigwedge_{i=1}^{\infty} \Omega_i = w_0$, *and that, for each i, at least one component* Δ_i *of* $f^{-1}(\Omega_i)$ *does not belong to* SO_{HD}, *then R does not belong to* O_{HD}.

Theorem 7. *Let R be a Riemann surface belonging to* O_{HD_n} $(1 \leqq n \leqq \infty)$, *let* Φ *be the covering surface of the w-plane generated by a non-constant single-valued meromorphic function f on R, and let* Φ_ρ *be a connected piece of* Φ *on* $|w-w_0|<\rho$. *If the area of* Φ_ρ *is finite, then the restriction* f_ρ *of f to the component* Δ_ρ *of* $f^{-1}(|w-w_0|<\rho)$ *corresponding to* Φ_ρ *is of type-Bl of* Δ_ρ. *So,* Φ_ρ *covers each point of* $|w-w_0|<\rho$ *the same times except at most a closed set of capacity zero and* Φ_ρ *is finitely sheeted.*

It is evident that this theorem implies Kuramochi's result (Theorem 12 in [5]).

References

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