152. *Note on Fundamental Exact Sequences in Homology and Cohomology for Non-normal Subgroups*

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The purpose of the present note is to observe that the fundamental exact sequences, or the exact sequences of Hochschild-Serre [4], in homology and cohomology of groups, which describe a certain relationship between homology or cohomology groups of a group, its normal subgroup, and the factor group, may be extended to the case of non-normal subgroups.

Thus, let G be a group and H a subgroup of G. With a (left) G-module M, Adamson [1] defines relative cohomology groups $H^n([G, H], M)$ on M, which in case H is normal in G turn out to coincide with the ordinary cohomology groups $H^n(G/H, M^H)$ of the factor group G/H, M^H being the submodule of M consisting of all elements of M left invariant by H. The relative cohomology groups $H^n([G, H], M)$ may be defined either in terms of the standard complex for $[G, H]$, as in [1], or more generally in terms of any $[G, H]$-projective resolution of the module Z of rational integers (i.e. a $(Z[G], Z[S])$-exact sequence $0\leftarrow Z\leftarrow X_0\leftarrow X_1\leftarrow\cdots$ of $Z[G]$-modules in which each X_i is $(Z[G], Z[S])$-projective), and may be expressed as $\mathrm{Ext}^n_{[G,H]}(Z, M)$ $(=\mathrm{Ext}^n_{(Z[G], Z[H])}(Z, M))$, in the terminology and notation of Hochschild [3]. Now, Adamson [1] proves that if here $H^m(U, M)=0$ for $m=1, \cdots, n-1$ $(n>0)$ and for every subgroup U of G which is an intersection of conjugates of H then the sequence

$$0\rightarrow H^n([G, H], M)\xrightarrow{\lambda} H^n(G, M)\xrightarrow{\rho} H^n(H, M)$$

is exact, where ρ is the ordinary restriction map and λ is the lifting (or inflation) map defined for instance by the natural map of the standard complex of G onto that of $[G, H]$. We contend that this exact sequence can be enlarged to a larger exact sequence which specializes to the exact sequence of Hochschild-Serre [4] in case H is normal in G. Thus, under the same assumption as above, $H^m(U, M)=0$ for $m=1, \cdots, n-1$ $(n>0)$ and for every subgroup U of G which is an intersection of conjugates of H, we have an exact sequence

$$(1)\qquad \begin{aligned}0\rightarrow H^n([G, H], M)&\xrightarrow{\lambda} H^n(G, M)\xrightarrow{\rho} H^n(H, M)^I\\ &\xrightarrow{\tau} H^{n+1}([G, H], M)\xrightarrow{\lambda} H^{n+1}(G, M),\end{aligned}$$

where the maps λ, ρ are as before, $H^n(H, M)^I$ is a certain subgroup of $H^n(H, M)$, and the map τ, transgression, is defined, similarly as in the case of normal H, as follows: a (standard) n-cocycle h of H, in M, is "transgressive" in case there are an n-cochain g of H and an $n+1$-cocycle f of $[G, H]$, both in M, such that $h=\rho g, \lambda f=(-1)^n\delta g$, and the cohomology class of h is mapped by τ to that of f; that the map is defined uniquely depends on our assumption on $H^m(U, M)$.

Next, the same (standard, say) complex for $[G, H]$ gives rise also to relative homology groups $H_n([G, H], M)$ in a G-module M. They may also be expressed as $\mathrm{Tor}_n^{[G, H]}(M, Z)$ $(=\mathrm{Tor}_n^{(Z[G], Z[H])}(M, Z))$, and in case H is normal in G they turn out to coincide with $H_n(G/H, M_H)$. Now, the result dual to the above is that if, with some G-module M, $H_m(U, M)=0$ for $m=1, \cdots, n-1$ $(n>0)$ and for every subgroup U of G which is an intersection of conjugates of H then we have an exact sequence

$$(2)\qquad \begin{aligned} 0 \leftarrow H_n([G, H], M) \xleftarrow{\varphi} H_n(G, M) \xleftarrow{\iota} H_n(H, M)_I \\ \xleftarrow{\tau} H_{n+1}([G, H], M) \xleftarrow{\varphi} H_{n+1}(G, M), \end{aligned}$$

where ι and φ are respectively the injection and the residuation (or deflation), $H_n(H, M)_I$ is a certain factor group of $H_n(H, M)$, and τ is defined (under our assumption) as follows: for every (standard) $n+1$-cycle h of $[G, H]$ in M there are an $n+1$-chain g of G and an n-cycle f of H, in M, such that $h=\varphi g$, $\iota f=(-1)^n\partial g$, and the original homology class of h is mapped to the class of the homology class of f.

In order to prove these, we adopt the inductive method of Adamson [1] (without "transgression") and Hattori [2] (with "transgression"), on establishing the case $n=1$ by direct verification. As to cohomology, the last is done rather readily. On the other hand, in the verification of the case $n=1$ of the exact sequence (2) in homology a key point lies in showing that for every 2-cycle h there are a 2-chain g of G and a 1-cycle f of H such that $h=\varphi g$, $\iota f=-\partial g$. Our proof, which includes a somewhat complicated (but quite concrete) construction proving this last, will be, together with precise descriptions of the groups $H^n(H, M)^I$ and $H_n(H, M)_I$, given shortly elsewhere.

Now, if H is of finite index in G, then cohomology groups $H^n([G, H], M)$ and homology groups $H_n([G, H], M)$ can both be defined for all rational integers $n=0, \pm 1, \pm 2, \cdots$ (Adamson [1] (for cohomology groups); Hochschild [3]). In order to have the same with $H^n(G, M)$, $H_n(G, M)$, $H^n(H, M)$, $H_n(H, M)$, we now assume that G itself is finite; here we have however $H^{-n-1}(G, M)=H_n(G, M)$, $H^{-n-1}(H, M)=H_n(H, M)$. Then there arise two new series of exact sequences, whose existence in the special case of H normal in G has been observed in the writer's previous note [5]. Thus, if, with some G-module M, $H^{-m}(U, M)=0$ for $m=0, \cdots, n-1$ $(n\geqq 0)$ and for every subgroup U of G which is an intersection of conjugates of H, then we have an exact sequence

$$(3)\qquad \begin{aligned} 0\leftarrow H^{-n}([G, H], M)\leftarrow H^{-n}(G, M)\leftarrow H^{-n}(H, M)_I \\ \leftarrow H^{-n-1}([G, H], M)\leftarrow H^{-n-1}(G, M). \end{aligned}$$

Dually, if $H_{-m}(U, M)=0$ for $m=0, \cdots, n-1$ $(n\geqq 0)$ and for every subgroup U of G which is an intersection of conjugates of H, then we have an exact sequence

$$(4)\qquad \begin{aligned} 0\rightarrow H_{-n}([G, H], M)\rightarrow H_{-n}(G, M)\rightarrow H_{-n}(H, M)^I \\ \rightarrow H_{-n-1}([G, H], M)\rightarrow H_{-n-1}(G, M). \end{aligned}$$

But the cases $n\geqq 2$ of these sequences turn out to be derived from the sequences (1), (2). This and other details of these sequences will also be discussed in a subsequent publication.

Not only that the inductive argument in our proof is borrowed from his paper [2], as is said above, Mr. A. Hattori is thanked by the writer for his kind collaboration in the context of the present work.

References

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