50. *Between-topology on a Distributive Lattice*

By Yatarō MATSUSHIMA
Gunma University, Maebashi

(Comm. by K. KUNUGI, M.J.A., May 7, 1959)

1. It is well known that the interval topology of a lattice L is defined by taking the closed intervals $[a)=\{x \mid x\geqq a\}$, $(a]=\{x \mid x\leqq a\}$ and $[a, b]=\{x \mid a\leqq x\leqq b\}$ as a sub-basis for closed sets. In [1–2] we have considered the concept of *B-covers* in lattices. For any two elements a and b of a lattice L, let

$B(a, b)=\{x \mid (a\smile x)\frown(b\smile x)=x=(a\frown x)\smile(b\frown x)\}$; then $B(a, b)$ is called the *B-cover* of a and b, and we write axb when $x\in B(a, b)$. Let $B^*(a,b)=\{x \mid abx\}$.

Now we shall define the *between-topology* on L as follows. By the *B-topology* (*B^*-topology*) of a lattice L, we mean that defined by taking the sets $B(a, b)$ ($B^*(a, b)$) as a sub-basis of closed sets.

In Theorem 1 we shall prove that the *B-topology* coincides with the interval topology in case L is a distributive lattice with O, I. It is shown in Theorem 2 that L_0 is a topological lattice in its *B^*-topology* when L_0 is a distributive lattice such that for any subset $B(a, b)$ of L_0, if $x, y\in B(a, b)$, then $a\frown x$ and $a\frown y$; $b\frown x$ and $b\frown y$ are comparable respectively.

E. S. Wolk [5] has defined that a subset X of a lattice L is *diverse* if and only if $x\in S$, $y\in S$, and $x\neq y$ imply that x and y are non-comparable. He showed that if L contains no infinite *diverse* set then L is a Hausdorff space in its interval topology.

Now we shall consider a distributive lattice L_0 with O, I satisfying the same assumption as in Theorem 2. Then in Theorem 3 we shall prove, by using the concept of the *B-covers* instead of that of *diverse* sets, that a certain type of L_0 is a Hausdorff space in its interval topology. This theorem is concerned with the Problem 23 of Birkhoff [3].

A *mob* is defined as a Hausdorff space with a continuous associative multiplication. In Theorem 4 we shall show that a distributive lattice L_0 with O, I such that $L_0=B(a_0, b_0)$ is a *mob* with the desired *kernel* $B(a, b)$ and with the multiplication defined as follows:

$xy=(a\smile x)\frown(b\smile y)$ for the fixed two elements a, b of L.

2. Lemma 1. *In a distributive lattice, $x\in B(a, b)$ if and only if $a\frown b\leqq x\leqq a\smile b$.*

Proof. This is proved in [1, Theorem 3].

Theorem 1. *In a distributive lattice L with O, I the B-topology* *coincides with the interval topology.*

Proof. By Lemma 1 $B(a,b)=[a\frown b, a\smile b]$. On the other hand, $[a)$, $(a]$ and $[a,b]$ are expressed by the sets of the type $B(a,b)$. Indeed $[a)=B(a,I)$, $(a]=B(O,a)$ and $[a,b]=B(a,b)$.

Lemma 2. *In a lattice L, $B^*(a,b)\ni x\smile y$ implies $x\in B^*(a\smile b, b)$ and $y\in B^*(a\smile b, b)$. $B^*(a,b)\ni x\frown y$ implies $x\in B^*(a\frown b, b)$ and $y\in B^*(a\frown b, b)$.*

Proof. Suppose that $x\bar{\in} B^*(a\smile b, b)$; then $(a\smile b)\frown(b\smile x)$ does not equal to b, and hence we have $(a\smile b)\frown(b\smile x)>b$. It follows that $ab(x\smile y)$ does not hold since $(a\smile b)\frown(b\smile x\smile y)\geqq(a\smile b)\frown(b\smile x)>b$. Thus either $x\bar{\in} B^*(a\smile b, b)$ or $y\bar{\in} B^*(a\smile b, b)$ implies $x\smile y\bar{\in} B^*(a,b)$, that is, $x\smile y\in B^*(a,b)$ implies $x\in B^*(a\smile b,b)$ and $y\in B^*(a\smile b,b)$. Dually $x\frown y\in B^*(a,b)$ implies $x\in B^*(a\frown b,b)$ and $y\in B^*(a\frown b,b)$.

Lemma 3. *In a distributive lattice L, if $x\in B^*(a\smile b,b)$ and $y\in B^*(a,b)$, then $x\smile y$ belongs to $B^*(a,b)$. Dually $x\in B^*(a\frown b,b)$ and $y\in B^*(a,b)$ imply $x\frown y\in B^*(a,b)$.*

Proof. Since L is distributive we have $(a\smile b)\frown x\leqq b\leqq a\smile b\smile x$, $a\frown y\leqq b\leqq a\smile y$ by Lemma 1. Then $b\leqq a\smile b\smile x\smile y=a\smile x\smile y$ since $a\smile y\geqq b$ as above. $b\geqq(a\smile b)\frown x\geqq a\frown x$, $b\geqq a\frown y$ imply $b\geqq(a\frown x)\smile(a\frown y)=a\frown(x\smile y)$. Thus we have $ab(x\smile y)$ by Lemma 1. Similarly we have the dual case.

Lemma 4. *Let L_0 be a distributive lattice satisfying the following condition* (A):

(A) *For any subset $B(a,b)$ in L_0, if $x,y\in B(a,b)$, then $a\frown x$ and $a\frown y$; $b\frown x$ and $b\frown y$ are comparable respectively.*

Then in L_0 $x,y\bar{\in} B^*(a,b)$ and $x,y\in B^*(a\smile b,b)$ imply $x\smile y\bar{\in} B^*(a,b)$ and $x\smile y\in B^*(a\smile b,b)$.

Proof. From $x\in B^*(a\smile b,b)$ and $x\bar{\in} B^*(a,b)$ we have $(a\frown b)\smile(b\frown x)<b$. Similarly $(a\frown b)\smile(b\frown y)<b$. Put $P=(a\frown b)\smile(b\frown x)$, $Q=(a\frown b)\smile(b\frown y)$. Then $(a\smile P)\frown(b\smile P)=(a\smile((a\frown b)\smile(b\frown x)))\frown(b\smile((a\frown b)\smile(b\frown x)))=(a\smile(b\frown x))\frown b=(a\frown b)\smile(b\frown x)=P$, and $(a\frown P)\smile(b\frown P)=(a\frown((a\frown b)\smile(b\frown x)))\smile(b\frown((a\frown b)\smile(b\frown x)))=(a\frown b)\smile(a\frown b\frown x)\smile(a\frown b)\smile(b\frown x)=(a\frown b)\smile(b\frown x)=P$ by distributive law, that is, $P\in B(a,b)$. Similarly $Q\in B(a,b)$. Hence $b\frown P=P$ and $b\frown Q=Q$ are comparable by the hypothesis. Accordingly we have $(a\frown b)\smile(b\frown(x\smile y))=(a\frown b)\smile(b\frown x)\smile(a\frown b)\smile(b\frown y)=P\smile Q<b$ since either $P\leqq Q<b$ or $Q\leqq P<b$, that is, $x\smile y\bar{\in} B^*(a,b)$. It is easily shown that $x\smile y\in B^*(a\smile b,b)$ from $x,y\in B^*(a\smile b,b)$.

Theorem 2. *Let L_0 be a distributive lattice satisfying the condition* (A), *then L_0 is a topological lattice in its B^*-topology.*

Proof. We shall prove the continuity of the join operation $x\smile y$. By Lemmas 2, 3 and 4 we have $x\smile y\bar{\in} B^*(a,b)$ if and only if one of the following conditions occurs:

(1) $x\bar{\in} B^*(a,b)$ and $y\bar{\in} B^*(a,b)$;

(2) $x \bar{\in} B^*(a \smile b, b)$;

(3) $y \bar{\in} B^*(a \smile b, b)$.

Hence we can prove the continuity of $x \smile y$. Similarly we can prove the continuity of $x \frown y$.

3. Definition. When $B^*(a, b) = b$ for some a in a lattice, we shall say that b is extreme for a, and denote this fact by $(a, b)E$. (a, b) is called an extreme pair when $B^*(a, b) = b$ and $B^*(b, a) = a$; in this case we shall write $(a, b)E_s$.

Lemma 5. *If a and a' are complemented, then $(a, a')E_s$.*

Proof. If $aa'x$, then $a' = (a \frown a') \smile (a' \frown x) = a' \frown x$, $a' = (a \smile a') \frown (a' \smile x) = a' \smile x$ from $a \frown a' = O$, $a \smile a' = I$, and hence $a' = x$. Similarly if $a'ax$, then $a = x$.

Lemma 6. *If $(a, b)E_s$, then a does not belong to any $B(a', b)$ such that $a \neq a'$ and a', b are non-comparable.*

Proof. If $a \in B(a', b)$, then $a'ab$, that is, $a' \in B^*(b, a)$, this contradicts $(a, b)E_s$.

If (a, b) is a non-comparable pair which is $(a, b)E_s$, $B(a, b)$ is called a *maximal extreme B-cover.*

Hereafter let L_0 be a distributive lattice with O, I satisfying the condition (A).

Lemma 7. *If L_0 consists of a finite number of maximal extreme B-covers and a chain, then L_0 is uniquely expressed as follows:*

(B) $L = \sum_{i=1}^{n} B(a_i, b_i) + C$,*) *where $B(a_i, b_i)$ are maximal extreme B-covers such* that *a_i, b_i are non-comparable, and C is a chain.*

Proof. It is proved from Lemmas 1, 5 and 6 and the condition (A).

Lemma 8. *If $B(a, b) \ni x$ in a distributive lattice L_0, then $B(a, b) = B(a, b \frown x) \smile B(b, a \smile x)$.*

Proof. If we take $y \in B(a, b \frown x) \smile B(b, a \smile x)$, then $a \frown b \leqq y \leqq a \smile b$, hence $y \in B(a, b)$. Conversely if we take $y \in B(a, b)$ then $b \frown y \geqq b \frown x$ implies $a \smile y \geqq a \smile x$, since $a \smile (b \frown y) \geqq a \smile (b \frown x)$, and $a \smile (b \frown y) = (a \smile b) \frown (a \smile y) = a \smile y$ and $a \smile (b \frown x) = a \smile x$. Similarly $a \smile x \geqq a \smile y$ implies $b \frown x \geqq b \frown y$. Accordingly we have either $b \frown y \geqq b \frown x$ or $a \smile x \geqq a \smile y$ since $b \frown x$ and $b \frown y$ are comparable in L_0. In the first case we have $a \smile b \geqq y \geqq b \frown x$, that is, $y \in B(b, a \smile x)$, and in the second case we have $a \smile x \geqq y \geqq a \frown b$, that is, $y \in B(a, b \frown x)$.

Lemma 9. *If $L_0 = B(a_0, b_0)$, where a_0, b_0 are non-comparable extreme pair, then L_0 is a Hausdorff space in its interval topology.*

Proof. Let a, b be distinct elements of L_0. From [4] we can prove that there is a covering of L_0 by means of a finite number of closed intervals such that no interval contains both a and b.

*) $\sum$, $+$ denote the set-theoretical unions.

(a) The case where a, b are non-comparable

Since L_0 is distributive and $a, b \in L_0 = B(a_0, b_0)$, we have either a_0ab or a_0ba by (A). We shall consider first the case a_0ab. Then abb_0 by [1, Lemma 3]. In this case L_0 is represented in the following form by Lemma 8.

$$L_0 = B(a_0, a) \smile B(b, b_0) \smile [a) \smile [b) \smile (a] \smile (b] \smile B(a, b). \qquad (1)$$

In (1), if $B(a, b) = \{a, b, a \frown b, a \smile b\}$, then we have $B(a, b) = [a, a \smile b] \smile [a \frown b, b]$, and if $B(a, b)$ contains x which is distinct from $a, b, a \frown b$ and $a \smile b$, then we have $B(a, b) = B(a, b \frown x) \smile B(b, a \smile x)$ by Lemma 8. Thus we have a covering of L_0 which has a desired form. In case a_0ba we can proceed similarly.

Consequently we have a covering of L_0 by means of a finite number of closed intervals such that no interval contains both a and b.

(b) The case where a, b are comparable

Suppose that $a > b$. If there is no x such that $a > x > b$, then one of the following coverings of L_0 is desired form by [2, § 4 (3)].

$$L_0 = B(a, b_0) \smile B(a_0, b) \smile [a) \smile (b],$$
$$L_0 = B(a, a_0) \smile B(b, b_0) \smile [a) \smile (b]. \qquad (2)$$

If there is x such that $a > x > b$, then put

$$L_0 = B(x, a_0) \smile B(x, b_0) \smile [x) \smile (x]. \qquad (3)$$

In (3), if $B(x, a_0)$ contains both a and b, then we shall divide it into parts as follows. If there is no y such that $a > y > x$, then let $B(a_0, x) = B(a, a_0) \smile B(b, a_0) \smile [b, x]$. In case there is y such that $a > y > x$, then let $B(a_0, x) = B(y, a_0) \smile (y]$, then we shall have the desired intervals.

If $B(x, b_0)$ contains both a and b in (3), we shall be able to divide it into the desired intervals similarly.

Theorem 3. *If L_0 is a distributive lattice with O, I and if it satisfies the conditions* (A) *and* (B), *then L_0 is a Hausdorff space in its interval topology.*

Proof. The theorem follows immediately from Lemmas 7 and 9.

4. Now we shall introduce a multiplication in a distributive lattice.

Definition. We shall define $xy = (a \smile x) \frown (b \smile y)$ for fixed two elements a, b of L.

Lemma 10. $x(yz) = (xy)z$ *in* L.

Proof. $x(yz) = (a \smile x) \frown (b \smile ((a \smile y) \frown (b \smile z))) = (a \smile x) \frown (a \smile b \smile y) \frown (b \smile z)$, $(xy)z = (a \smile ((a \smile x) \frown (b \smile y))) \frown (b \smile z) = (a \smile x) \frown (a \smile b \smile y) \frown (b \smile z)$.

Lemma 11. *If $x \in B(a, b)$ and $y \in L$, then we have*

(1) $xx = x$,

(2) $xy \in B(a, b)$, $yx \in B(a, b)$.

Proof. Since (1) is immediate from the definition, we shall prove (2).

$(a \smile xy) \frown (b \smile xy) = (a \smile ((a \smile x) \frown (b \smile y))) \frown (b \smile ((a \smile x) \frown (b \smile y))) = (a \smile x)$

$\frown(a\smile b\smile y)\frown(a\smile b\smile x)\frown(b\smile y)=(a\smile x)\frown(b\smile y)=xy$; similarly $(a\frown xy)\smile(b\frown xy)=xy$. Thus L is a semigroup with the kernel $B(a,b)$.

Theorem 4. *Let L_0 be a distributive lattice with O, I such that $L_0=B(a_0,b_0)$ satisfies the condition* (A), *where a_0, b_0 are non-comparable extreme pair. Then the multiplication $xy=(a\smile x)\frown(b\smile y)$ is continuous in its interval topology, that is, L_0 is a mob which has the desired kernel $B(a,b)$.*

Proof. Suppose that $xy=(a\smile x)\frown(b\smile y)$ belongs to some B-cover $B(c,d)$. Since $a,b,c,d\in B(a_0,b_0)$ we shall prove the continuity for xy in case a_0ab, acb, adb, and acd. Then a_0ab, adb imply a_0ad by [1, Lemma 4] and a_0ad, acd imply a_0cd by [1, Lemma 8]. Hence cdb_0 by [1, Lemma 3]. By [1, Lemma 2] we have $a_0\smile d\geqq a_0\smile c$, $b_0\smile c\geqq b_0\smile d$ and $a_0\frown c\geqq a_0\frown d$ and $b_0\frown d\geqq b_0\frown c$.

From $a_0\smile(b_0\frown c)=a_0\smile c$, $a_0\smile(b_0\frown d)=a_0\smile d$, $(a_0\smile d)\frown(b_0\smile c)=c\smile d$, $(a_0\smile c)\frown(b_0\smile d)=c\frown d$ and [2, §4 (3)], if $x\bar{\in}B(b_0\frown c, a_0\smile d)$, then we have either $a_0\smile x>a_0\smile d$ or $a_0\smile x<a_0\smile c$.

If $a_0\smile x>a_0\smile d$, then we have $a\smile x>a\smile d$ and $(a_0\smile x)\frown b_0>b_0\frown d$ by [2, §4 (3)], hence $xy\in B(a_0,b_0)-(a_0\smile d]$ since $a\smile x$, $b\in B(a_0,b_0)-(a_0\smile d]$, and if $a_0\smile x<a_0\smile c$, then we have $xy\bar{\in}B(c,d)$ similarly. $y\bar{\in}B(a_0\frown d, b_0\smile c)$ implies $xy\bar{\in}B(c,d)$ in the same way. Hence $x\bar{\in}B(b_0\frown c, a_0\smile d)$ or $y\bar{\in}B(a_0\frown d, b_0\smile c)$ implies $xy\bar{\in}B(c,d)$.

Conversely if $x\in B(b_0\frown c, a_0\smile d)$ and $y\in B(a_0\frown d, b_0\smile c)$, then $xy\in B(c,d)$, that is, $xy\bar{\in}B(c,d)$ implies $x\bar{\in}B(b_0\frown c, a_0\smile d)$ or $y\bar{\in}B(a_0\frown d, b_0\smile c)$. This completes the proof.

Corollary. *Let L_0 be a distributive lattice with O, I satisfying the conditions* (A) *and* (B); *then L_0 is a mob.*

References

[1] Y. Matsushima: On the B-covers in lattices, Proc. Japan Acad., **32**, 549-553 (1956).

[2] Y. Matsushima: The geometry of lattices by B-covers, Proc. Japan Acad., **33**, 328-332 (1957).

[3] G. Birkhoff: Lattice Theory, rev. ed., New York (1948).

[4] E. H. Northam: The interval topology of a lattice, Proc. Amer. Math. Soc., **4**, 824-827 (1953).

[5] E. S. Wolk: Order-compatible topology on a partially ordered set, Proc. Amer. Math. Soc., **9**, 524-529 (1958).