49. *On the Extensions of Finite Factors. II*

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Since extensions of a continuous finite factor A are closely related with extensions of the group K of all inner automorphisms of A [2], some fundamentals of the cohomology theory of groups reflect upon constructions of extended factors. In this paper we shall show that the effectiveness of a group G of automorphism classes for the construction of extended factors is decided by the fact that a three-dimensional cochain associated with G is coboundary or not. In general, the group K has no central element other than 1 and so, by a proposition of group extensions, the extension of K by G is uniquely determined within equivalences. On the other hand we shall define an equivalence relation in factors extended by G analogously to the one for extended groups and then show that the equivalent classes of extensions of A by G are one-to-one correspondent to the second cohomology group $H^2(G, Z)$, where Z is the unit circle in the complex plane and G is assumed to act on Z trivially.

1. We use the same notations as in [2] as possible. By A we mean a continuous finite factor acting on a separable Hilbert space and by $\tilde{\mathfrak{A}}$ the group of all $*$-automorphisms of A. Denote by K the group of all inner automorphisms of A. K is a normal subgroup of $\tilde{\mathfrak{A}}$. Put $\mathfrak{A}$ the quotient group $\tilde{\mathfrak{A}}/K$. We take up an enumerable subgroup G of $\mathfrak{A}$. We call G *a group of automorphism classes.* For every element $\alpha\in G$ we choice a representative $\bar{\alpha}$ in the coset α of the quotient $\tilde{\mathfrak{A}}/K$, then for every α and β there occurs $m_{\alpha,\beta}\in K$ such that $\bar{\alpha}\cdot\bar{\beta}=\overline{\alpha\beta}\cdot m_{\alpha,\beta}$. This satisfies relations:

$$(1) \qquad (k^{\alpha})^{\beta} = (k^{\alpha\beta})^{m_{\alpha,\beta}} \qquad \text{for } k\in K$$

$$(2) \qquad m_{\alpha,\beta\gamma}m_{\beta,\gamma} = m_{\alpha\beta,\gamma}m^{\gamma}_{\alpha,\beta}$$

where $k^{\alpha}=\bar{\alpha}^{-1}k\bar{\alpha}$ and $k^{m}=m^{-1}km$ $(m\in K)$. We call such a system $\{m_{\alpha,\beta}\}$ *a factor set of inner automorphisms of A.* If a factor set $\{m_{\alpha,\beta}\}$ satisfies $m_{\alpha,\alpha^{-1}}=1$ for every α, it is *normalized.* In this paper we consider only such a group for which normalized factor sets exist. For a factor set $\{m_{\alpha,\beta}\}$, we get an extension $\boldsymbol{K}$ of the group K by G, which we show by $\boldsymbol{K}=(K, G, m_{\alpha,\beta})$ [1, 2].

Let $\boldsymbol{K}^{(1)}=(K, G, m^{(1)}_{\alpha,\beta})$ and $\boldsymbol{K}^{(2)}=(K, G, m^{(2)}_{\alpha,\beta})$ be two extensions of a group K by a group G with respect to different factor sets $\{m^{(1)}_{\alpha,\beta}\}$ and $\{m^{(2)}_{\alpha,\beta}\}$ respectively. If there is an isomorphism between $\boldsymbol{K}^{(1)}$ and $\boldsymbol{K}^{(2)}$ satisfying

(i) $1\otimes k\in \boldsymbol{K}^{(1)}\leftrightarrow 1\otimes k\in \boldsymbol{K}^{(2)}$ (i.e. the identity mapping on K)

(ii) $\alpha\otimes k\in \boldsymbol{K}^{(1)}\leftrightarrow \alpha\otimes h\in \boldsymbol{K}^{(2)}$,

these extensions are said *equivalent*. It is known that $\boldsymbol{K}^{(1)}$ is equivalent to $\boldsymbol{K}^{(2)}$ if and only if there exists $n_\alpha\in K$ $(\alpha\in G)$ such that

(3) $$\overline{\alpha}_{(2)}=\overline{\alpha}_{(1)}n_\alpha, \qquad m^{(2)}_{\alpha,\beta}=n^{-1}_{\alpha\beta}m^{(1)}_{\alpha,\beta}n^{\beta}_{\alpha}n_\beta$$

(where $n^\beta_\alpha=\overline{\beta}^{-1}_{(1)}n_\alpha\overline{\beta}_{(1)}$). Hence we say that two factor sets are equivalent with each other if they are connected by the relations (3).

For an inner automorphism k of A, there is a unitary operator $v\in A$ such that $x^k=v^*xv$ $(x\in A)$, but this v is not determined uniquely. Every unitary operator w satisfying $w^*v=\chi\cdot 1$ (χ is a complex number) induces the same inner automorphism k.

Now let $v_{\alpha,\beta}$ be a unitary operator in A which induces the inner automorphism $m_{\alpha,\beta}$, then by (2) we get in general

(4) $$v_{\alpha,\beta\gamma}v_{\beta,\gamma}\cdot\chi(\alpha,\beta,\gamma)=v_{\alpha\beta,\gamma}v^{\gamma}_{\alpha,\beta}$$

(where $\chi(\alpha,\beta,\gamma)$ is a complex number such that $|\chi(\alpha,\beta,\gamma)|=1$). If $v_{\alpha,\beta}$ is suitably chosen to satisfy

(5) $$\chi(\alpha,\beta,\gamma)=1, \quad v_{1,1}=1, \; v_{\alpha,\alpha^{-1}}=v_{\alpha^{-1},\alpha}=\lambda_\alpha 1$$

for every α,β,γ (λ_α is a complex number), we call $\{v_{\alpha,\beta}\}$ *a normalized factor set of unitary operators of* A. Using such a normalized factor set, we are able to construct an extension $\boldsymbol{A}=(A, G, v_{\alpha,\beta})$ of the factor A, which admits a group of inner automorphisms isomorphic to $\boldsymbol{K}$ $=(K, G, m_{\alpha,\beta})$ [2].

2. For any case does there appear a normalized factor set of unitary operators? We discuss it in this section.

Let Z be the unit circle of the complex plane and we assume that G acts trivially on Z, i.e. $z^\alpha=z$ for every $z\in Z$ and $\alpha\in G$. Put $C^3(G,Z)$ the group of 3-dimensional cocycles on G with values in Z*) and by $B^3(G,Z)$ we show the subgroup of coboundaries of 2-dimensional cochains, that is, the collection of cocycles $\varphi(\alpha,\beta,\gamma)$ which may be expressed as

$$\varphi(\alpha,\beta,\gamma)=\frac{f(\alpha\beta,\gamma)f(\alpha,\beta)}{f(\alpha,\beta\gamma)f(\beta,\gamma)}$$

by a Z-valued function $f(\alpha,\beta)$ on $G\times G$. The quotient group $C^3(G,Z)/B^3(G,Z)$ is usually called the *third cohomology group.*

LEMMA 1. *The element of the third cohomology group which contains the cocycle* $\chi(\alpha,\beta,\gamma)$ *in* (4) *depends upon only* G *and is independent from the choice of representatives* $\overline{\alpha}$ *and unitary operators* $v_{\alpha,\beta}$.

Proof. Let $\overline{\alpha}$ be a representative of $\alpha\in G$ and $\overline{\alpha}'$ be another one then there is an element n_α in K such that $\overline{\alpha}'=\overline{\alpha}n_\alpha$ and we get

*) A Z-valued function $\varphi(\alpha,\beta,\gamma)$ on $G\times G\times G$ is a cocycle if it satisfies $\varphi(\beta,\gamma,\delta)\cdot\varphi(\alpha,\beta\gamma,\delta)\;\varphi(\alpha,\beta,\gamma)/\varphi(\alpha\beta,\gamma,\delta)\;\varphi(\alpha,\beta,\gamma\delta)=1$ for every $\alpha,\beta,\gamma,\delta$. $\chi(\alpha,\beta,\gamma)$ in (4) is a cocycle.

$$\overline{\alpha}' \cdot \overline{\beta}' = \overline{\alpha} n_\alpha \cdot \overline{\beta} n_\beta = \overline{\alpha\beta} \cdot m_{\alpha,\beta} n_\alpha^\beta n_\beta = \overline{\alpha\beta}' \cdot n_{\alpha\beta}^{-1} m_{\alpha,\beta} n_\alpha^\beta n_\beta.$$

We show by $a^{\alpha'}$ the image of a by the automorphism $\overline{\alpha}'$. Put w_γ the unitary operator which induces the inner automorphism n_γ then $w_{\alpha\beta}^* v_{\alpha,\beta} w_\alpha^\beta w_\beta$ induces $n_{\alpha\beta}^{-1} m_{\alpha,\beta} n_\alpha^\beta n_\beta$. Put $v'_{\alpha,\beta} = w_{\alpha\beta}^* v_{\alpha,\beta} w_\alpha^\beta w_\beta$ then $w_{\alpha\beta} v'_{\alpha,\beta} = v_{\alpha,\beta} w_\alpha^\beta w_\beta = v_{\alpha,\beta} w_\beta w_\alpha^{\beta'}$ and

$$\begin{aligned} & w_{\alpha\beta\gamma} v'_{\alpha\beta,\gamma} [v'_{\alpha,\beta}]^{\gamma'} = v_{\alpha\beta,\gamma} w_\gamma w_{\alpha\beta}^{\gamma'} [v'_{\alpha,\beta}]^{\gamma'} = v_{\alpha\beta,\gamma} w_\gamma [w_{\alpha\beta} v'_{\alpha,\beta}]^{\gamma'} \\ &= v_{\alpha\beta,\gamma} w_\gamma [v_{\alpha,\beta} w_\beta w_\alpha^{\beta'}]^{\gamma'} = v_{\alpha\beta,\gamma} w_\gamma v_{\alpha,\beta}^{\gamma'} [w_\beta w_\alpha^{\beta'}]^{\gamma'} \\ &= v_{\alpha\beta,\gamma} v_{\alpha,\beta}^{\gamma} w_\gamma [w_\beta w_\alpha^{\beta'}]^{\gamma'} = v_{\alpha,\beta\gamma} v_{\beta,\gamma} \chi(\alpha, \beta, \gamma) w_\gamma w_\beta^{\gamma'} (w_\alpha^{\beta'})^{\gamma'} \\ &= v_{\alpha,\beta\gamma} w_{\beta\gamma} v'_{\beta,\gamma} (w_\alpha^{\beta'})^{\gamma'} \chi(\alpha, \beta, \gamma) = v_{\alpha,\beta\gamma} w_{\beta\gamma} w_\alpha^{(\beta\gamma)'} v'_{\beta,\gamma} \chi(\alpha, \beta, \gamma) \\ &= w_{\alpha\beta\gamma} v'_{\alpha,\beta\gamma} v'_{\beta,\gamma} \chi(\alpha, \beta, \gamma). \end{aligned}$$

This means $v'_{\alpha,\beta\gamma} v'^{\gamma'}_{\alpha,\beta} = v'_{\alpha,\beta\gamma} v'_{\beta,\gamma} \cdot \chi(\alpha, \beta, \gamma)$. That is, we know that if we employ a suitably chosen system of unitary operators $\{v_{\alpha,\beta}\}$ and $\{v'_{\alpha,\beta}\}$, there appears the same cocycle $\chi(\alpha, \beta, \gamma)$ corresponding to different systems of representatives $\{\overline{\alpha}\}$ and $\{\overline{\alpha}'\}$.

Next we fix a system of automorphisms $\{\overline{\alpha}\}$, then inner automorphisms $m_{\alpha,\beta}$ are determined consequently. If $\{v_{\alpha,\beta}\}$ is a system of unitary operators such that each $v_{\alpha,\beta}$ induces the inner automorphism $m_{\alpha,\beta}$ of A and $\{v'_{\alpha,\beta}\}$ is another such a system, there occurs a 2-dimensional cochain $f(\alpha, \beta)$ with values in Z to satisfy $v'_{\alpha,\beta} = f(\alpha, \beta) v_{\alpha,\beta}$. Then by

$$v_{\alpha,\beta\gamma} v_{\beta,\gamma} \chi(\alpha, \beta, \gamma) = v_{\alpha\beta,\gamma} v_{\alpha,\beta}^\gamma, \quad v'_{\alpha,\beta\gamma} v'_{\beta,\gamma} \chi'(\alpha, \beta, \gamma) = v'_{\alpha\beta,\gamma} v'^{\gamma}_{\alpha,\beta},$$

we get

$$\chi'(\alpha, \beta, \gamma) = \frac{f(\alpha\beta, \gamma) f(\alpha, \beta)}{f(\alpha, \beta\gamma) f(\beta, \gamma)} \chi(\alpha, \beta, \gamma).$$

This means that the cocycles $\chi(\alpha, \beta, \gamma)$ and $\chi'(\alpha, \beta, \gamma)$ are cohomologous. Combining with the first step of proof we get the conclusion.

THEOREM 1. *A normalized factor set of unitary operators is associated to a group G if and only if the cocycle $\chi(\alpha, \beta, \gamma)$ in* (4) *for arbitrarily chosen $\{v_{\alpha,\beta}\}$ is a coboundary.*

Proof. If a normalized factor set $\{v_{\alpha,\beta}\}$ is associated for suitably chosen $\{\overline{\alpha}\}$ and $\{m_{\alpha,\beta}\}$, then $\chi(\alpha, \beta, \gamma) \equiv 1$. Thus if we put $f(\alpha, \beta) = 1$ for every pair α, β

$$\chi(\alpha, \beta, \gamma) \equiv \frac{f(\alpha\beta, \gamma) f(\alpha, \beta)}{f(\alpha, \beta\gamma) f(\beta, \gamma)}$$

Hence by Lemma 1, the condition is necessary.

On the contrary, we assume that $\chi(\alpha, \beta, \gamma)$ satisfies the condition stated in the theorem. Put $v'_{\alpha,\beta} = 1/f(\alpha, \beta) \cdot v_{\alpha,\beta}$ then

$$\chi'(\alpha, \beta, \gamma) = \frac{f(\alpha, \beta\gamma) f(\beta, \gamma)}{f(\alpha\beta, \gamma) f(\alpha, \beta)} \chi(\alpha, \beta, \gamma) \equiv 1.$$

Hence $v'_{\alpha,\beta\gamma} v'_{\beta,\gamma} = v'_{\alpha\beta,\gamma} v'^{\gamma}_{\alpha,\beta}$, especially $v'_{\alpha,1} = v'_{1,1}$, $v'_{1,\alpha} = v'^{\alpha}_{1,1}$. We may assume $v'_{1,1} = 1$. Furthermore, since

$$\frac{f(\alpha, 1)f(\alpha^{-1}, \alpha)}{f(1, \alpha)f(\alpha, \alpha^{-1})}\chi(\alpha, \alpha^{-1}, \alpha)=1,$$

$$v_{\alpha,1}v_{\alpha^{-1},\alpha}\frac{f(1, \alpha)f(\alpha, \alpha^{-1})}{f(\alpha, 1)f(\alpha^{-1}, \alpha)}=v_{1,\alpha}v^{\alpha}_{\alpha,\alpha^{-1}},$$

$$v'_{\alpha,1}v'_{\alpha^{-1},\alpha}=v'_{1,\alpha}v'^{\alpha}_{\alpha,\alpha^{-1}} \text{ i.e. } \quad v'_{\alpha^{-1},\alpha}=v'_{\alpha,\alpha^{-1}}=\lambda_\alpha 1$$

because $m_{\alpha,\alpha^{-1}}=m_{\alpha^{-1},\alpha}=1$. Hence $\{v'_{\alpha,\beta}\}$ is a normalized factor set of unitary operators satisfying the required condition.

3. Hereafter we treat only groups G for which the condition of Theorem 1 is satisfied. Hence we may assume the existence of both extensions of A and of K by G. We show the uniqueness of the extension of K within equivalences.

LEMMA 2. *The group K of all inner automorphisms of a finite factor A has no central element other than the identity.*

Proof. Let k be a central element of K, that is, $kh=hk$ for every $h\in K$. Put u_k, u_h the unitary operators which induces k, h respectively. Then $u_h^*u_k^*xu_ku_h=u_k^*u_h^*xu_hu_k$ for every $x\in A$. Hence we get $u_hu_ku_h^*u_k^*=\lambda_h 1$ (λ_h is a complex number) and so $u_k^*u_hu_k=\lambda_hu_h$. For the trace τ of A, $\tau(u_h)=\tau(u_k^*u_hu_k)=\lambda_h\tau(u_h)$. Thus if $\tau(u_h)\neq 0$, $\lambda_h=1$ i.e. $u_hu_k=u_ku_h$.

Now let $e\in A$ be a projection such that $\tau(e)\neq\frac{1}{2}$, then $2e-1$ is a unitary operator and $\tau(2e-1)\neq 0$. Hence $(2e-1)u_k=u_k(2e-1)$ i.e. $e=u_k^*eu_k$. If $\tau(e)=\frac{1}{2}$, there are two projections e_1, e_2 satisfying $e=e_1+e_2$, $e_1\sim e_2$, $e_1\perp e_2$, $e_1=u_k^*e_1u_k$ and $e_2=u_k^*e_2u_k$. That is $u_k^*eu_k=e$, and so k preserves invariant every projection of A. This means that k is the identity automorphism.

COROLLARY. *The extension of K is determined uniquely by G within equivalences.*

This follows from Lemma 2 and the general theory of group extensions [1, § 52 Extensions of group without centre].

4. By the above corollary, we know that each normalized factor set of inner automorphisms associated to G is always equivalent to another one. Thus two factor sets $\{m^{(1)}_{\alpha,\beta}\}$ and $\{m^{(2)}_{\alpha,\beta}\}$ such that

$$\bar{\alpha}\cdot\bar{\beta}=\overline{\alpha\beta}\cdot m^{(1)}_{\alpha,\beta}, \qquad \bar{\alpha}'\cdot\bar{\beta}'=\overline{\alpha\beta}'\cdot m^{(2)}_{\alpha,\beta}$$

$$m^{(1)}_{\alpha,\beta\gamma}\cdot m^{(1)}_{\beta,\gamma}=m^{(1)}_{\alpha\beta,\gamma}\cdot m^{(1)\gamma}_{\alpha,\beta}, \qquad m^{(2)}_{\alpha,\beta\gamma}m^{(2)}_{\beta,\gamma}=m^{(2)}_{\alpha\beta,\gamma}m^{(2)\gamma'}_{\alpha,\beta},$$

are connected by

$$\bar{\alpha}'=\bar{\alpha}n_\alpha \ (n_\alpha\in K), \qquad m^{(2)}_{\alpha,\beta}=n^{-1}_{\alpha\beta}m^{(1)}_{\alpha,\beta}n^{\beta}_{\alpha}n_\alpha.$$

Denote by $\{v^{(1)}_{\alpha,\beta}\}$ and $\{v^{(2)}_{\alpha,\beta}\}$ the normalized factor sets of unitary operators corresponding $\{m^{(1)}_{\alpha,\beta}\}$ and $\{m^{(2)}_{\alpha,\beta}\}$ respectively and by w_α the unitary operator which induces n_α, then we get the relation

$$v^{(2)}_{\alpha,\beta}=\psi(\alpha, \beta)w^*_{\alpha\beta}v^{(1)}_{\alpha,\beta}w^{\beta}_{\alpha}w_\beta, \tag{6}$$

where $\psi(\alpha, \beta)$ is a two-dimensional cochain on G with values in Z.

LEMMA 3. *$\psi(\alpha, \beta)$ is a two-dimensional cocycle, that is, it holds the equality $\psi(\alpha, \beta\gamma)\psi(\beta, \gamma)=\psi(\alpha\beta, \gamma)\psi(\alpha, \beta)$ for every α, β, γ.*

Proof. By the assumption

$$v^{(2)}_{\alpha,\beta\gamma}=\psi(\alpha, \beta\gamma)w^*_{\alpha\beta\gamma}v^{(1)}_{\alpha,\beta\gamma}w^{\beta\gamma}_{\alpha}w_{\beta\gamma}$$
$$v^{(2)}_{\beta,\gamma}=\psi(\beta, \gamma)w^*_{\beta\gamma}v^{(1)}_{\beta,\gamma}w^{\gamma}_{\beta}w_{\gamma}$$
$$v^{(2)}_{\alpha\beta,\gamma}=\psi(\alpha\beta, \gamma)w^*_{\alpha\beta\gamma}v^{(1)}_{\alpha\beta,\gamma}w^{\gamma}_{\alpha\beta}w_{\gamma}$$
$$v^{(2)\gamma\prime}_{\alpha,\beta}=w^*_{\gamma}v^{(2)\gamma}_{\alpha,\beta}w_{\gamma}=w^*_{\gamma}(\psi(\alpha, \beta)w^*_{\alpha\beta}v^{(1)}_{\alpha,\beta}w^{\beta}_{\alpha}w_{\beta})^{\gamma}w_{\gamma}.$$

Since $v^{(1)}_{\alpha,\beta\gamma}v^{(1)}_{\beta,\gamma}=v^{(1)}_{\alpha\beta,\gamma}v^{(1)\gamma}_{\alpha,\beta}$, $v^{(2)}_{\alpha,\beta\gamma}v^{(2)}_{\beta,\gamma}=v^{(2)}_{\alpha\beta,\gamma}v^{(2)\gamma\prime}_{\alpha,\beta}$, we get

$$\psi(\alpha, \beta\gamma)\psi(\beta, \gamma)=\psi(\alpha\beta, \gamma)\psi(\alpha, \beta). \qquad \text{q.e.d.}$$

In the equality $\psi(\alpha, \beta\gamma)\psi(\beta, \gamma)=\psi(\alpha\beta, \gamma)\psi(\alpha, \beta)$ putting $\alpha=1$, $\beta=1$ and $\gamma=\alpha$, we get $\psi(1, \alpha)=\psi(1, 1)$, for $\beta=1$, $\gamma=1$ or $\beta=\alpha^{-1}$, $\gamma=\alpha$, $\psi(\alpha, 1)=\psi(1, 1)$ or $\psi(\alpha^{-1}, \alpha)=\psi(\alpha, \alpha^{-1})$ respectively. Thus if $\{v_{\alpha,\beta}\}$ is a normalized factor set of unitary operators and $\psi(\alpha, \beta)$ is a cocycle satisfying $\psi(1, 1)=1$, then $\{\psi(\alpha, \beta)v_{\alpha,\beta}\}$ is a normalized factor set of unitary operators again. We notice too that the element of the second cohomology group $H^2(G, Z)$ which contains the cocycle $\psi(\alpha, \beta)$ is independent of the choice of w_α, because if $w'_\alpha=\rho(\alpha)w_\alpha$ $\rho(\alpha)\in Z)$, the corresponding $\psi'(\alpha, \beta)$ determined for w'_α satisfies $\psi(\alpha, \beta)=\psi'(\alpha, \beta)\cdot \rho(\alpha)\rho(\beta)/\rho(\alpha\beta)$.

Similarly as for extensions of a group we define an equivalence relation for extensions of a factor. Let $\boldsymbol{A}^{(1)}=(A, G, v^{(1)}_{\alpha,\beta})$ and $\boldsymbol{A}^{(2)}=(A, G, v^{(2)}_{\alpha,\beta})$ be two extensions of a factor A by a group G with respect to factor sets of unitary operators $\{v^{(1)}_{\alpha,\beta}\}$ and $\{v^{(2)}_{\alpha,\beta}\}$ respectively and $D^{(1)}$, $D^{(2)}$ be the algebraic crossed products for each case [2]. If there is a $*$-isomorphism between $D^{(1)}$ and $D^{(2)}$ satisfying

(i) $1\otimes a\in D^{(1)}\leftrightarrow 1\otimes a\in D^{(2)}$ i.e. the identity mapping on A,

(ii) $\alpha\otimes a\in D^{(1)}\leftrightarrow \alpha\otimes b\in D^{(2)}$

(iii) $\tau^{(1)}((\alpha\otimes a)(\beta\otimes b)^*)=\tau^{(2)}((\alpha\otimes c)(\beta\otimes d)^*)$ if $\alpha\otimes a\leftrightarrow\alpha\otimes c$ and $\beta\otimes b\leftrightarrow\beta\otimes d$,

(where $\tau^{(1)}$, $\tau^{(2)}$ are the traces of $\boldsymbol{A}^{(1)}$, $\boldsymbol{A}^{(2)}$ respectively), we call $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ are *equivalent extensions* of the factor A.

LEMMA 4. *Two extensions $\boldsymbol{A}^{(1)}$, $\boldsymbol{A}^{(2)}$ of a continuons finite factor A by a group G with respect to normalized factor sets $\{v^{(1)}_{\alpha,\beta}\}$ and $\{v^{(2)}_{\alpha,\beta}\}$ are equivalent if and only if the cocycle $\psi(\alpha, \beta)$ in (6) is a coboundary.*

Proof. Necessity. If $\boldsymbol{A}^{(2)}$ is equivalent to $\boldsymbol{A}^{(1)}$, $\alpha\otimes 1\in\boldsymbol{A}^{(2)}$ is mapped to $\alpha\otimes u_\alpha\in\boldsymbol{A}^{(1)}$. Since $\alpha\otimes 1$ is a unitary operator in $\boldsymbol{A}^{(2)}$ $\alpha\otimes u_\alpha$ is a unitary operator and so u_α is a unitary operator in A. $\alpha\otimes a_\alpha=(\alpha\otimes 1)(1\otimes a_\alpha)$ is mapped to $(\alpha\otimes u_\alpha)(1\otimes a_\alpha)=\alpha\otimes u_\alpha a_\alpha$. Hence

$$(\alpha\otimes 1)(\beta\otimes 1)\rightarrow(\alpha\otimes u_\alpha)(\beta\otimes u_\beta)=\alpha\beta\otimes v^{(1)}_{\alpha,\beta}u^{\beta}_{\alpha}u_{\beta}.$$

On the other hand

$$(\alpha\otimes 1)(\beta\otimes 1)=\alpha\beta\otimes v^{(2)}_{\alpha,\beta}\rightarrow\alpha\beta\otimes u_{\alpha\beta}v^{(2)}_{\alpha,\beta}.$$

Thus we get

$$v^{(2)}_{\alpha,\beta}=u^*_{\alpha\beta}v^{(1)}_{\alpha,\beta}u^{\beta}_{\alpha}u_{\beta}.$$

This shows $\psi(\alpha,\beta)\equiv 1$. Therefore the cocycle $\psi(\alpha,\beta)$ which appears in (6) for other different choice of $\{w_\alpha\}$ is a coboundary.

Sufficiency. If $\psi(\alpha,\beta)$ is a coboundary,

$$v^{(2)}_{\alpha,\beta}=(\rho(\alpha)\rho(\beta)/\rho(\alpha\beta))u^*_{\alpha\beta}v^{(1)}_{\alpha,\beta}u^\beta_\alpha u_\beta.$$

We replace $\rho(\alpha)u_\alpha$, $\rho(\beta)u_\beta$, $\rho(\alpha\beta)u_{\alpha\beta}$ with u_α, u_β, $u_{\alpha\beta}$ respectively, then the above equality changes to

$$v^{(2)}_{\alpha,\beta}=u^*_{\alpha\beta}v^{(1)}_{\alpha,\beta}u^\beta_\alpha u_\beta.$$

We may define a linear mapping from $D^{(2)}$ into $D^{(1)}$ such that

$$\alpha\otimes a\in D^{(2)}\to\alpha\otimes u_\alpha a\in D^{(1)}.$$

Clearly this is one-to-one and maps $D^{(2)}$ onto $D^{(1)}$. Since $u_1=1$, the mapping is identity on A and preserves multiplication and $*$-operation.

$$(\alpha\otimes a_\alpha)(\beta\otimes b_\beta)=\alpha\beta\otimes v^{(2)}_{\alpha,\beta}a^{\beta'}_\alpha b_\beta\to\alpha\beta\otimes u_{\alpha\beta}v^{(2)}_{\alpha,\beta}u^*_\beta a^\beta_\alpha u_\beta b_\beta=\alpha\beta\otimes v^{(1)}_{\alpha,\beta}u^\beta_\alpha a^\beta_\alpha u_\beta b_\beta.$$

On the other hand $\alpha\otimes a_\alpha\to\alpha\otimes u_\alpha a_\alpha$, $\beta\otimes b_\beta\to\beta\otimes u_\beta b_\beta$ and

$$(\alpha\otimes u_\alpha a_\alpha)(\beta\otimes u_\beta b_\beta)=\alpha\beta\otimes v^{(1)}_{\alpha,\beta}(u_\alpha a_\alpha)^\beta u_\beta b_\beta=\alpha\beta\otimes v^{(1)}_{\alpha,\beta}u^\beta_\alpha a^\beta_\alpha u_\beta b_\beta.$$

Hence multiplication is preserved.

$$(\alpha\otimes a)^*=\alpha^{-1}\otimes a^{(\alpha^{-1})'*}v^{(2)*}_{\alpha,\alpha^{-1}}\to\alpha^{-1}\otimes u_{\alpha^{-1}}a^{(\alpha^{-1})'*}v^{(2)*}_{\alpha,\alpha^{-1}}=\alpha^{-1}\otimes a^{\alpha^{-1}*}u_{\alpha^{-1}}v^{(2)*}_{\alpha,\alpha^{-1}},$$

$\alpha\otimes a\to\alpha\otimes u_\alpha a$ and $(\alpha\otimes u_\alpha a)^*=\alpha^{-1}\otimes(u_\alpha a)^{\alpha^{-1}*}v^{(1)*}_{\alpha,\alpha^{-1}}=\alpha^{-1}\otimes a^{\alpha^{-1}*}u_\alpha^{\alpha^{-1}*}v^{(1)*}_{\alpha,\alpha^{-1}}$. Since $v^{(2)*}_{\alpha,\alpha^{-1}}=u^*_{\alpha^{-1}}u_\alpha^{\alpha^{-1}*}v^{(1)*}_{\alpha,\alpha^{-1}}u_1$ and $u_1=1$, the mapping preserves $*$-operation. The condition (iii) concerning the traces remains valid doubtless. Therefore $\boldsymbol{A}^{(2)}$ is equivalent to $\boldsymbol{A}^{(1)}$. q.e.d.

Since the factor group $H^2(G,Z)=C^2(G,Z)/B^2(G,Z)$ is known as the second cohomology group, we get

THEOREM 2. *If it is possible to construct extensions of a continuous finite factor A by a group G of automorphism classes, then the equivalent classes of those extensions correspond one-to-one to the second cohomology group $H^2(G,Z)$.*

References

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