45. *Note on a Theorem for Dimension*

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1. Recently, K. Nagami has proved the following theorem [4]:

Let X and Y be metric spaces and f a closed continuous mapping of X onto Y. If $f^{-1}(y)$ consists of exactly $k(<\infty)$ points for every point $y\in Y$ and $\dim X\leqq 0$, then we have $\dim Y\leqq 0$.

In the present note, as an extension of this theorem, we shall prove the following theorem:

Theorem. *Let f be a closed continuous mapping of a metric space X onto a topological space Y such that for each point y of Y the inverse image $f^{-1}(y)$ consists of exactly $k(<\infty)$* points, *then we have*

$$dim\, X = dim\, Y.$$

To prove the theorem, we use some lemmas:

Lemma 1 (K. Morita [2]). *In order that a T_1-space X be metrizable it is necessary and sufficient that there exist a countable collection $\{\mathfrak{F}_j\}$ of locally finite closed covering of X satisfying the condition:*

For any neighborhood U of any point x of X there exists some j such that $S(x, \mathfrak{F}_j)\subset U$.

Lemma 2 (K. Morita and S. Hanai [3], A. H. Stone [5]). *Let f be a closed continuous mapping of a metric space X onto a topological space Y. In order that Y be metrizable it is necessary and sufficient that the boundary $\mathfrak{B}f^{-1}(y)$ of the inverse image $f^{-1}(y)$ be compact for every point y of Y.*

2. **Proof of the theorem.** Let us put $f^{-1}(y)=\{x_i(y)\mid i=1, 2,\cdots, k\}$ for every point y of Y. By Lemma 1 there exist a countable number $\{\mathfrak{F}_j\}$ of locally finite closed coverings of X such that for some integers j_i and some indices $\alpha_i\in\Omega_{j_i}$ we have

$$F_{j_i\alpha_i}\ni x_i(y), \qquad i=1, 2,\cdots, k$$

and

$$F_{j_i\alpha_i}\frown F_{j_l\alpha_l}=\phi, \qquad i, j=1, 2,\cdots, k, \qquad i\neq l,$$

where we put $\mathfrak{F}_j=\{F_{j\alpha}|\alpha\in\Omega_j\}, \qquad j=1, 2,\cdots$.

Let us put $\bigcap_{i=1}^{k} f(F_{j_i\alpha_i})=W_y$. As f is a closed mapping, W_y is a closed subset of Y and contains y. If we denote by f_1 the partial mapping f whose domain is $F_{j_1\alpha_1}\frown f^{-1}(W_y)$, and whose range is W_y, then f_1 is a homeomorphism from $F_{j_1\alpha_1}\frown f^{-1}(W_y)$ onto W_y. Hence we have

$$\dim W_y = \dim (F_{j_1\alpha_1} \frown f^{-1}(W_y)).$$

We put $\mathfrak{W} = \{W_y | y \in Y\}$, then $\mathfrak{W}$ is a closed covering of Y. By the assumption of the theorem and the property of $\{\mathfrak{F}_j\}$, $f(\mathfrak{F}_j) = \{f(\mathfrak{F}_{j\alpha}) | \alpha \in \Omega_j\}$ is a locally finite closed covering of Y.

Let us denote the totality of all the sets which consist of k distinct positive integers by $\Gamma_1, \Gamma_2, \cdots$. If we put $\mathfrak{Z}_p = \bigwedge_{j \in \Gamma_p} f(\mathfrak{F}_j)$, then $\mathfrak{Z}_p$ is a locally finite closed covering of Y and

$$\mathfrak{W} = \bigcup_{p=1}^{\infty} (\mathfrak{W} \frown \mathfrak{Z}_p).$$

As $\mathfrak{W} \frown \mathfrak{Z}_p$ is a locally finite closed system and by Lemma 2 Y is a metrizable space, we have

$$\dim Y_p = \dim \bigcup_{W_y \in \mathfrak{W} \frown \mathfrak{Z}_p} W_y \leqq \dim X.$$

Here, we put $Y_p = \bigcup_{W_y \in \mathfrak{W} \frown \mathfrak{Z}_p} W_y$. As Y_p is, of course, a closed subset of Y, we have

$$\dim Y = \dim \bigcup_{p=1}^{\infty} Y_p \leqq \dim X.$$

Next, we shall show $\dim X \leqq \dim Y$. By the construction we have $\dim f^{-1}(W_y) = \dim W_y$. For each integer p, $f^{-1}(\mathfrak{W} \frown \mathfrak{Z}_p) = \{f^{-1}(W_y) | W_y \in \mathfrak{W} \frown \mathfrak{Z}_p\}$ is a locally finite closed system of X. Hence $X_p = \bigcup_{W_y \in \mathfrak{W} \frown \mathfrak{Z}_p} f^{-1}(W_y)$ is a closed subset of X and $\dim X_p \leqq \dim Y$. Consequently, we have $\dim X = \dim \bigcup_{p=1}^{\infty} X_p \leqq \dim Y$. q.e.d.

Corollary. *Let f be a closed continuous mapping of a metric space X onto a topological space Y such that for every point y of Y the inverse image $f^{-1}(y)$ is finite, then for any finite m, we have*

$$\dim\{y \,|\, |f^{-1}(y)| = m\} \leqq \dim X.$$

References

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