61. On Locally Bounded Functions

By Kiyoshi ISÉKI
Kobe University

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Recently some characterisations of countably compact spaces and pseudo-compact spaces were obtained by A. Appert [1], E. Hewitt [3], J. Kersten [4] and J. Marík [5].

Following G. Aquaro [2], we shall define locally boundedness of functions, and we shall prove some results on locally boundedness.

Let S be a topological space, and let $f(x)$ be a real valued finite function (not necessary continuous) on S. A function $f(x)$ is said to be *locally bounded at a point* x_0, if there is a neighbourhood V of x_0 such that $f(x)$ is bounded on V. A function $f(x)$ is said to be *locally bounded* if $f(x)$ is locally bounded at every point of S.

It is clear that any continuous function on S is locally bounded. We shall show the following

Proposition 1. 1) *If S is countably compact, then any locally bounded function on S is bounded.*

2) *If every locally bounded function on S is bounded, then S is pseudo-compact.*

Proof. The second part of Proposition 1 is clear. To prove the first part, we shall suppose that there is a locally bounded and unbounded function $f(x)$. For every positive integer n, the set $A_n=\{x \mid |f(x)|\geq n\}$ is not empty, and the sequence of sets $\{A_n\}$ is decreasing. Therefore $\{\bar{A}_n\}_{n=1,2,\dots}$ is a decreasing sequence of closed sets. Hence $\bigcap_{n=1}^{\infty}\bar{A}_n$ is not empty. Let $x_0\in\bigcap_{n=1}^{\infty}\bar{A}_n$, since $f(x)$ is locally bounded at the point x_0, there is a neighbourhood V of x_0 such that $f(x)$ is bounded on V. Hence, $|f(x)|\leq N$ on V for an integer N. On the other hand, by $x_0\in\bigcap_{n=1}^{\infty}\bar{A}_n$, there is a point x' in V such that $|f(x')|>N$. This completes the proof.

Corollary 1. *The following conditions of a normal space S are equivalent*:

1) *S is countably compact.*

2) *S is pseudo-compact.*

3) *Any locally bounded function on S is bounded.*

Corollary 2. *A metric space is compact if and only if every locally bounded function on it is bounded.*

A sequence $\{f_n\}$ of functions on S is said to *converge to f quasi* *uniformly in the sense of Aquaro*, if, for every $\varepsilon > 0$ and positive integer ν, there is a locally bounded integer valued function $\nu(\varepsilon, x)$ on S such that $\nu \leq \nu(\varepsilon, x)$ and $|f_{\nu(\varepsilon,x)}(x) - f(x)| < \varepsilon$.

A topological space S is *metacompact* if and only if every open covering of S has a point finite refinement. Then we have the following

Proposition 2. *If a sequence of continuous functions on metacompact space S converges to* 0, *then the covergence is quasi-uniformly in the sense of Aquaro.*

Proof. Let $\{f_n(x)\}$ be a sequence of continuous functions such that $f_n(x) \to 0$ on S.

Given $\varepsilon > 0$ and positive integer N, let $O_n = \{x \mid |f_n(x)| < \varepsilon\}$, $n = N, N+1, \cdots$. Since $f_n(x) \to 0$, $\{O_n\}_{n=N,N+1,\dots}$ is an open covering of S. By the metacompactness of S, we can find a point-finite open covering $\omega = \{U_\alpha\}$ of S such that ω is a point finite refinement of $\{O_n\}$. Let $x \in S$, then there is finite set of all $\alpha_1, \cdots, \alpha_k$ such that $x \in U_{\alpha_i}$ $(i=1, 2, \cdots, k)$. For each U_{α_i}, we can take the first O_{n_i} containing U_{α_i}, and let $\nu(\varepsilon, x) = \text{Min } (n_1, \cdots, n_k)$. For every point x, the integral valued function $\nu(\varepsilon, x)$ is well-defined and $\nu(\varepsilon, x) \geq N$. It is clear the $|f_{\nu(\varepsilon,x)}(x)| < \varepsilon$ for every point x of S. For any point x of S, take all open sets U_{α_i} $(i=1, 2, \cdots, k)$ containing x, and let $V = \bigcap_{i=1}^{k} U_{\alpha_i}$. For any y of V, the family of all open sets containing y contains U_{α_i} $(i=1, 2, \cdots, n)$. Therefore we have $\nu(\varepsilon, y) \leq \min (n_1, \cdots, n_n) = \nu(\varepsilon, x)$. Hence $\nu(\varepsilon, x)$ is locally bounded. This completes the proof.

Proposition 2 implies the following

Corollary 3. *If a sequence of continuous functions on a metacompact* (*paracompact, countably paracompact*) *space converges to a continuous function, then the convergence is quasi-uniform in the sense of Aquaro on the space.*

References

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