83. A Remark on Monotone Solutions of Differential Equations

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In this Note, we shall consider a system of differential equations:

$$\frac{dx}{dt}=P(t)x+Q(t) \tag{1}$$

where, $x=(x_1, x_2, \cdots, x_n)$ is a column vector function of t, and $P(t)=(P_{ij}(t))$, $Q(t)=(Q_{ij}(t))$ are $n\times n$-matrices. Suppose that these functions of t are defined in some interval $[a, +\infty)$ (a is finite or $-\infty$). Any solution

$$x(t)=\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

of the system (1) is called to be *monotone for* $t\to\infty$, if for some T, every function $x_i(t)$ is monotone on the interval $[T, +\infty)$. A matrix function $P(t)=(P_{ij}(t))$ is said to be *integrable on an interval* $[a, +\infty)$, if every function $|P_{ij}(t)|$ is integrable on the interval.

Then we have the following

Proposition. If matrices $P(t)$ and $Q(t)$ of the differential equation (1) *are integrable on an interval* $[a, +\infty)$, *then any monotone solution $x(t)$ for $t\to+\infty$ is bounded on the interval and* $\lim_{t\to\infty} x(t)$ *exists.*

Such a type of Proposition was discussed by B. P. Demidobitch [1]. Applying his method, we shall prove Proposition directly.

To prove Proposition, we shall suppose that $n=1$ for simplicity. Let $x(t)$ be a monotone solution of (1) for an interval $[a, +\infty)$. Since $P(t)$, $Q(t)$ are integrable, for any $\varepsilon>0$ $(1>\varepsilon)$, there is a large number T, and then we have

$$\int_{t_1}^{t_2}|P(t)|\,dt<\varepsilon, \quad \int_{t_1}^{t_2}|Q(t)|\,dt<\varepsilon$$

for $t_1, t_2\geq T$. From (1), we can write

$$x(t)=x(T)+\int_T^t P(u)x(u)du+\int_T^t Q(u)du.$$

By the mean theorem of integral calculus, there is an α such that $T\leq\alpha\leq t$ and

$$x(t)=x(T)+x(T)\int_T^\alpha P(u)du+x(t)\int_\alpha^t P(u)du+\int_T^t Q(u)du.$$

Therefore, we have

$$\left(1-\int_{\alpha}^{t} P(u)du\right)x(t)=\left(1+\int_{T}^{\alpha} P(u)du\right)x(T)+\int_{T}^{t} Q(u)du$$

and, since ε is less than 1, $1-\int_{\alpha}^{t}|P(u)|du \neq 0$. Hence, for $t \geq T$,

$$|x(t)| \leq 1\Big/\left(1-\int_{\alpha}^{t}|P(u)|du\right)\left[\left(1+\int_{T}^{\alpha}|P(u)|du\right)|x(T)|+\int_{T}^{t}|Q(u)|du\right]$$

$$\leq \frac{1}{1-\varepsilon}[(1+\varepsilon)|x(T)|+\varepsilon]$$

and we have the boundedness of $x(t)$, and $\lim_{t\to+\infty} x(t)$ exists. This completes the proof.

Reference

[1] B. P. Demidobitch: On the boundedness of monotone solutions of a system of linear differential equations, Uspehi Mate-Nauk, **12**, 143–146 (1957).