78. *Cauchy Integral on Riemann Surfaces*

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In recent times many authors treated the linear functional methods in the theory of functions of a complex variable. In these studies it is of great use to study the behaviour of the considered function on the boundary of the domain.

In this paper we will study the boundary behaviour of the Cauchy integral on a Riemann surface.

1. Let $\mathfrak{B}$ be a compact or non-compact subregion with finite relative boundaries on a non-compact Riemann surface. We can recognize the existence of a differential $dN(\mathfrak{P}_\zeta, \mathfrak{P}_z)=A(\zeta, z)d\zeta$ with the following properties (Elementardifferential of $\mathfrak{R}$ in the terminology of Behnke-Stein).[1]

$A(\zeta, z)$ is defined in a cylinder region $\mathfrak{R}_{\mathfrak{P}_\zeta}\times\mathfrak{R}_{\mathfrak{P}_z}$, and

1. $A(\zeta, z)$ is meromorphic with respect to both arguments.
2. If $\mathfrak{P}_\zeta\neq\mathfrak{P}_z$, $dN(\mathfrak{P}_\zeta, \mathfrak{P}_z)$ is finite.
3. If $\mathfrak{P}_\zeta=\mathfrak{P}_z$, $dN(\mathfrak{P}_\zeta, \mathfrak{P}_z)$ has a pole with residue 1 at $\mathfrak{P}_z$.

When $f(\mathfrak{P}_\zeta)$ is a continuous function on the relative boundary Γ of $\mathfrak{B}$ (where $\Gamma=\bigcup_{j=1}^{p}\gamma_j$, and γ_j is an analytic Jordan curve), and $dN(\mathfrak{P}_\zeta, \mathfrak{P}_z)$ is an Elementardifferential of $\mathfrak{R}$, the integral $F(\mathfrak{P}_z)=\frac{1}{2\pi i}\int_\Gamma f(\mathfrak{P}_\zeta)\cdot dN(\mathfrak{P}_\zeta, \mathfrak{P}_z)$ may be called the Cauchy integral of $\mathfrak{B}$. Each of the integral $\frac{1}{2\pi i}\int_{\gamma_j} f_j(\mathfrak{P}_\zeta)dN(\mathfrak{P}_\zeta, \mathfrak{P}_z)$, where $f_j(\mathfrak{P}_\zeta)$ is the restriction of $f(\mathfrak{P}_\zeta)$ on γ_j, is called the j-th component of $F(\mathfrak{P}_z)$ and denoted by $F_j(\mathfrak{P}_z)$.

In the sequel it is fundamentally important that for each j $(j=1, 2,\cdots, p)$ there exist

1) a strip $\mathfrak{R}_j$ of $\mathfrak{R}$ containing γ_j,

2) an annulus $r_1<x<r_2$ in the complex plane with $r_1<1<r_2$,

3) a one-to-one conformal mapping of the strip $\mathfrak{R}_j$ into the annulus such that γ_j is mapped on the circumference of the unit circle.

The strip and the associated mapping are not in any way unique, but in our study it will sometimes be convenient to fix one of them and denote it by $\mathfrak{R}_j$ and $\mathfrak{P}_\zeta=\lambda_j(t)$. Given a function $f(\mathfrak{P}_z)$ on $\mathfrak{R}$ and

1) Cf. H. Behnke und F. Sommer: Theorie der analytischen Funktionen einer komplexen Veränderlichen, 555–559 (1955).

a function $g(\mathfrak{P}_\zeta)$ on Γ, we shall say that $f(\mathfrak{P}_z)$ tends uniformly to the boundary value $g(\mathfrak{P}_\zeta)$ on Γ from the interior or the exterior of $\mathfrak{B}$, if x approaches non-tangentially to t from $|x|<1$ or from $|x|>1$, and $\lim_{x\to t-} f(\lambda_j(x))=g(\lambda_j(t))$, $\lim_{x\to t+} f(\lambda_j(x))=g(\lambda_j(t))$ respectively. Further we shall define a function $f(\mathfrak{P}_\zeta)$ on Γ as being an element of $H(\mu)$ class, if for each j, $f_j(\mathfrak{P}_\zeta)$ satisfies locally the Hölder-condition of order $\mu(0<\mu\leqq 1)$ on Γ. So $f_j(\lambda_j(t))$ belongs to $H(\mu)$ on the unit circle.

If $\mathfrak{P}_z\notin\gamma_j$, the function $F_j(\mathfrak{P}_z)$ is represented by the form $\frac{1}{2\pi i}\int_{\gamma_j} f(\zeta)A(\zeta, z)\,d\zeta$ where $A(\zeta, z)$ is an analytic function of both arguments. We can easily see $F_j(\mathfrak{P}_z)$ is analytic outside of γ_j. Suppose now $\mathfrak{P}_{\zeta_0}\in\gamma_j$, $\mathfrak{P}_\zeta=\lambda_j(t)$ and $\mathfrak{P}_z=\lambda_j(x)$, $\varphi_j(t)=f_j(\lambda_j(t))$. The integral $F_j(\mathfrak{P}_z)$ is represented in $\mathfrak{N}_j$,

$$\frac{1}{2\pi i}\int_{|t|=1}\varphi_j(t)\left\{\frac{1}{t-x}+h(t, x)\right\}dt$$

where $h(t, x)$ is an analytic function of both variables. Thus we find that $F_j(\mathfrak{P}_z)$ is represented in $\mathfrak{N}_j$

$$\frac{1}{2\pi i}\int_{|t|=1}\varphi_j(t)\frac{dt}{t-x}+g_j(x)$$

where $g_j(x)$ is locally holomorphic function on $|x|=1$. Since $\frac{1}{2\pi i}\int_{|t|=1}\frac{\varphi_j(t)\,dt}{t-x}$ is a Cauchy type integral in $|x|<1$, we put

$$\Phi_j^+(t_0)=\frac{1}{2}\varphi_j(t_0)+\frac{1}{2\pi i}\int_{|t|=1}\frac{\varphi_j(t)}{t-t_0}\,dt$$

$$\Phi_j^-(t_0)=-\frac{1}{2}\varphi_j(t_0)+\frac{1}{2\pi i}\int_{|t|=1}\frac{\varphi_j(t)}{t-t_0}\,dt.$$

The function $\Phi_j(x)=\frac{1}{2\pi i}\int_{|t|=1}\frac{\varphi_j(t)}{t-x}\,dt$ tends uniformly to the limit $\Phi_j^+(t_0)$ or $\Phi_j^-(t_0)$ whether x approaches non-tangentially to t_0 in $|x|<1$ or in $|x|>1$ respectively.

So the integral $F_j(\mathfrak{P}_z)$ tends uniformly to the limit $F_j^+(\mathfrak{P}_{\zeta_0})$ or $F_j^-(\mathfrak{P}_{\zeta_0})$ whether $\mathfrak{P}_z$ approaches non-tangentially to $\mathfrak{P}_0$ in $\mathfrak{N}_j\frown\mathfrak{B}$ or in $\mathfrak{N}_j\frown\mathfrak{B}^*$ where $\mathfrak{B}^*$ is the complement of $\mathfrak{B}$ and

$$F_j^+(\mathfrak{P}_{\zeta_0})=\frac{1}{2}f(\mathfrak{P}_{\zeta_0})+\frac{1}{2\pi i}\int_{\gamma_j} f_j(\mathfrak{P}_\zeta)\,dN(\mathfrak{P}_\zeta, \mathfrak{P}_{\zeta_0})$$

$$F_j^-(\mathfrak{P}_{\zeta_0})=-\frac{1}{2}f(\mathfrak{P}_{\zeta_0})+\frac{1}{2\pi i}\int_{\gamma_j} f_j(\mathfrak{P}_\zeta)\,dN(\mathfrak{P}_\zeta, \mathfrak{P}_{\zeta_0}).$$

Now we arrive at the

Theorem 1. *If the function $f(\mathfrak{P}_z)$ satisfies locally the Hölder-condition on Γ and $\mathfrak{P}_{\zeta_0}\in\gamma_j$, the function $F(\mathfrak{P}_z)$ tends uniformly to the limit $F^+(\mathfrak{P}_{\zeta_0})$ or $F^-(\mathfrak{P}_{\zeta_0})$ whether $\mathfrak{P}_z$ approaches non-tangentially* *to* $\mathfrak{P}_{\zeta_0}$ *in* $\mathfrak{N}_j \frown \mathfrak{B}$ *or in* $\mathfrak{N}_j \frown \mathfrak{B}^*$ *where* $\mathfrak{B}^*$ *is the complement of* $\mathfrak{B}$ *and*

$$F^+(\mathfrak{P}_{\zeta_0}) = \frac{1}{2} f(\mathfrak{P}_{\zeta_0}) + \frac{1}{2\pi i} \int_{\Gamma} f(\mathfrak{P}_\zeta)\, dN(\mathfrak{P}_\zeta, \mathfrak{P}_{\zeta_0})^{2)}$$

$$F^-(\mathfrak{P}_{\zeta_0}) = -\frac{1}{2} f(\mathfrak{P}_{\zeta_0}) + \frac{1}{2\pi i} \int_{\Gamma} f(\mathfrak{P}_\zeta)\, dN(\mathfrak{P}_\zeta, \mathfrak{P}_{\zeta_0}).$$

Thus we have $F^+(\mathfrak{P}_{\zeta_0}) - F^-(\mathfrak{P}_{\zeta_0}) = f(\mathfrak{P}_{\zeta_0})$ *on* Γ.

2. The space $\bigcup_{0<\mu\leq 1} H(\mu)$ will be denoted by $\mathcal{H}$. We consider two operators on $\mathcal{H}$.

$$L^+ f = \frac{1}{2} f(\mathfrak{P}_{\zeta_0}) + \frac{1}{2\pi i} \int f(\mathfrak{P}_\zeta)\, dN(\mathfrak{P}_\zeta, \mathfrak{P}_{\zeta_0})$$

$$L^- f = \frac{1}{2} f(\mathfrak{P}_{\zeta_0}) - \frac{1}{2\pi i} \int f(\mathfrak{P}_\zeta)\, dN(\mathfrak{P}_\zeta, \mathfrak{P}_{\zeta_0}).$$

The functions L^+f and L^-f are holomorphically extensible into $\mathfrak{B}$ and into each component of $\mathfrak{B}^*$ respectively. The extended function will again be denoted by L^+f and L^-f. Then on the boundary of $\mathfrak{B}$

$$L^+f + L^-f = f. \tag{1}$$

We shall call this decomposition of f into sectionally holomorphic functions *a Privalov-Plemely decomposition of f with respect to an Elementardifferential dN.* The operators L^+, L^- have the following properties.

1) L^+, L^- are linear operators on $\mathcal{H}$.

2) $L^+f \in \mathcal{H}$, $L^-f \in \mathcal{H}$ (due to Plemely-Privalov's theorem).[3]

3) L^+ and L^- are continuous; i.e. if $f_n \rightrightarrows f$ $(\in \mathcal{H})$ on Γ, $L^+f_n \rightrightarrows L^+f$ in $\mathfrak{B}$ and $L^-f_n \rightrightarrows L^-f$ in each component of $\mathfrak{B}^*$.

4) L^+ and L^- are commutative; i.e. $L^+(L^-f) = L^-(L^+f)$. In fact

$$L^+(L^-f) = L^+f - L^+(L^+f)$$
$$L^-(L^+f) = L^+f - L^+(L^+f).$$

If $d\omega = \sigma(\mathfrak{P}_\zeta) d\mathfrak{P}_\zeta$ is locally holomorphic differential on Γ.

We put $L^+d\omega = L^+\sigma(\mathfrak{P}_\zeta)d\mathfrak{P}_\zeta$, $L^-d\omega = L^-\sigma(\mathfrak{P}_\zeta)d\mathfrak{P}_\zeta$, then $L^+d\omega$ and $L^-d\omega$ are holomorphically extensible into $\mathfrak{B}$ and into each component of $\mathfrak{B}^*$ respectively.

On the boundary of $\mathfrak{B}$, we again have the relation

$$L^+d\omega + L^-d\omega = d\omega. \tag{2}$$

Now a function $f(\mathfrak{P}_\zeta)$ (or a differential $d\omega(\mathfrak{P}_\zeta)$) is described as a boundary function of $\mathfrak{B}$ and $\mathfrak{B}^*$ (or a boundary differential of $\mathfrak{B}$ or $\mathfrak{B}^*$) with respect to the relation (1) (or (2)). We shortly say that a function $f(\mathfrak{P}_\zeta)$ belongs to $C(\mathfrak{B})$, if and only if $L^-f=0$ and belongs to

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2) The integral $\frac{1}{2\pi i}\int_{\Gamma} f_j(\mathfrak{P}_\zeta)\, dN(\mathfrak{P}_\zeta, \mathfrak{P}_{\zeta_0})$ must be described as the principal value of the Cauchy integral but we will denote it by the usual symbol $\int$ instead of (v.p) $\int$.

3) Cf. N. I. Muskelishivili: Singular Integral Equations, 46–47 (1946) (translation from the Russian edited by J. R. N. Radok).

$\mathfrak{C}(\mathfrak{B}^*)$ if and only if $L^+f=0$. For the differential $d\omega$, we denote the associated classes by $\mathfrak{C}'(\mathfrak{B})$ and $\mathfrak{C}'(\mathfrak{B}^*)$.

Thus we have

Theorem 2. *The function (or the differential) L^+f belongs to $\mathfrak{C}(\mathfrak{B})$ (or $\mathfrak{C}'(\mathfrak{B})$) if and only if L^-f belongs to $\mathfrak{C}(\mathfrak{B}^*)$ (or $\mathfrak{C}'(\mathfrak{B}^*)$).*

Proof. The necessary and sufficient condition of $L^+f\in\mathfrak{C}(\mathfrak{B})$ is $L^-(L^+f)=0$. From 4), we have $L^+(L^-f)=0$. This completes the proof.

If $\bar{\mathfrak{B}}^*$ is compact, the Cauchy's theorem successfully be applied to L^-f and we can conclude that $L^-f\in\mathfrak{C}(\mathfrak{B}^*)$.

Corollary. *If $\bar{\mathfrak{B}}^*$ is compact, L^+f belongs to $\mathfrak{C}(\mathfrak{B})$.*

3. The converse of the Cauchy's theorem due to W. Rudin reads as follows.[4]

Let $\mathfrak{B}$ be a relatively compact region bounded by a finite number of analytic Jordan curves $\Gamma=\bigcup_{j=1}^{p}\gamma_j$ in the complex plane and g be a bounded measurable function defined on Γ such that $\int_\Gamma g(\zeta)d\omega=0$ for all differential $d\omega$ analytic in the closure of $\mathfrak{B}$. Then g represents p.p. the boundary value of a function analytic in $\mathfrak{B}$. Concerning this theorem we state

Theorem 3. *If $\bar{\mathfrak{B}}^*$ is compact, the necessary and sufficient condition to be $f\in\mathfrak{C}(\mathfrak{B})$ is $\int_\Gamma f(\mathfrak{P}_\zeta)L^+d\omega=0$ for every differential locally holomorphic on Γ.*

Lemma. *Let $\mathfrak{B}^*$ be a relatively compact region bounded by a finite number of analytic Jordan curves Γ. If $f(\mathfrak{P}_\zeta)$ be a function analytic in $\mathfrak{B}^*$, continuous on $\bar{\mathfrak{B}}^*$ and $d\omega$ be a differential locally holomorphic on Γ. When $\int_\Gamma fd\omega=0$ for every $d\omega$, we have $f(\mathfrak{P}_\zeta)=0$ on Γ.*

In fact by Cauchy's theorem $f(\mathfrak{P}_z)=\frac{1}{2\pi i}\int_\Gamma f(\mathfrak{P}_\zeta)\,dN(\mathfrak{P}_\zeta,\mathfrak{P}_z)$. This concludes $f(\mathfrak{P}_z)=0$ from our orthogonal assumption. The continuity of $f(\mathfrak{P}_z)$ on $\bar{\mathfrak{B}}^*$ implies $f(\mathfrak{P}_\zeta)=0$ on Γ.

Proof of the theorem

The relation

$$\int_\Gamma fL^+d\omega=\int_\Gamma L^-f\cdot d\omega \tag{3}$$

is well known.[5] Then if $f\in\mathfrak{C}(\mathfrak{B})$, $L^-f=0$ by the definition of $\mathfrak{C}(\mathfrak{B})$

4) Cf. W. Rudin: Analytic function of class H_p, Trans. Amer. Math. Soc., **78** (1955); see also A. Read: A converse of Cauchy's theorem and application to extremal problem, Acta Math., **100**, 1-22 (1958).

5) Cf. Muskelishivili's monography, 122-123, loc. cit.

class. From (3) we have $\int_\Gamma fL^+d\omega=0$ for every $d\omega$. Conversely if $\int_\Gamma fL^+d\omega=0$ for every $d\omega$, we have also by (3) $\int_\Gamma L^-f\cdot d\omega=0$ for every $d\omega$. We have by the lemma $L^-f=0$ on Γ.

4. Now we shall call the operator $Lf=L^+f-L^-f$ on $\mathcal{H}$ a Cauchy operator. The Cauchy operator Lf is also linear continuous. Further referring to the fact $L^+f\in\mathcal{C}(\mathfrak{B})$ and the relation (1), we have the mutually equivalent relations $L^-(L^+f)=0$, $L^+(L^+f)=L^+f$, $L^-(L^-f)=L^-f$.

So the operator is an involution. Certainly $L(Lf)=L^+(L^+f)-2L^+(L^-f)+L^-(L^-f)=L^+f+L^-f=f$. This is the well-known inversion formula of the Cauchy integral. Next we must ask what are the characteristic properties of the Cauchy operator. Recently the characterization of the Cauchy operator by means of Elementardifferential has solved by H. Tietz in the space of locally holomorphic function on Γ.[7] His reasoning can also successfully be applied in our case. Let us consider a pair of linear operators L^+, L^- defined on $\mathcal{H}$.

If (L^+, L^-) satisfies the following conditions.

(a) $L^+\mathcal{H}\subset\mathcal{A}(\bar{\mathfrak{B}})$.[6]

(b) $L^-\mathcal{H}=\mathcal{A}(\mathfrak{B}^*)$.

(c) $L^+f+L^-f=f$.

(d) $L^-(L^-f)=L^-f$.

(e) If $f_n\Rightarrow f$ on Γ, $L^+f_n\Rightarrow L^+f$ in $\mathfrak{B}$ and $L^-f_n\Rightarrow L^-f$ in $\mathfrak{B}^*$.

(f) Let (L^+, L^-) be given. For every subregion $\mathfrak{B}'$ of $\mathfrak{B}$ with the relative boundary Γ' homologous to Γ such that each γ'_j is contained in $\mathfrak{N}_j$, there exists a prolongation (L_1^+, L_1^-) of (L^+, L^-), i.e. $L^+f=L_1^+f$ in $\mathfrak{B}'$ for every f which is locally holomorphic on $\overline{\mathfrak{B}'^*-\mathfrak{B}^*}$.

(g) $\int_\Gamma L^+f\cdot L^+d\omega=0$ for every $f\in\mathcal{H}$ and locally holomorphic differentials $d\omega$ on Γ. Then there exists an Elementardifferential $dN_0(\mathfrak{P}_\zeta, \mathfrak{P}_z)$ of $\mathfrak{N}$ uniquely determined by (L^+, L^-) such that L^+ and L^- are represented by a Plemely-Privalov's decomposition with respect to $dN_0(\mathfrak{P}_\zeta, \mathfrak{P}_z)$.

6) $\mathcal{A}(\bar{\mathfrak{B}})$ or $(\mathcal{A}(\bar{\mathfrak{B}}^*))$ is the space of functions which are holomorphic in $\mathfrak{B}$ (or in $\mathfrak{B}^*$) and its boundary value function belongs to $\mathcal{H}$.

7) Cf. H. Tietz: Funktionen mit Cauchyscher Integraldarstellung auf nicht kompakten Gebieten Riemannscher Flächen, Ann. Acad. Sci. Fenn., ser. A.I., Mathematica, no. 250/36, 1–9 (1958).