108. *Remarks on Pseudo-resolvents and Infinitesimal Generators of Semi-groups*

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Let X be a Banach space and $E(X)$ the algebra of all bounded linear operators on X to X. As is well known, a linear operator A in X is the infinitesimal generator of a semi-group $\{U(t)\}$, $0<t<\infty$, $U(t)\in E(X)$, if i) A is densely defined, ii) the resolvent $(\lambda I-A)^{-1}\in E(X)$ exists for sufficiently large real λ and $||(\lambda I-A)^{-1}||=O(\lambda^{-1})$ for $\lambda\to+\infty$ and iii) certain additional conditions are satisfied according to the types of semi-groups considered.[1]

The object of the present note is to point out that i) *is a consequence of* ii), *provided that the underlying space X is locally sequentially weakly compact* (abbr. l.s.w.c.). In particular this is the case if X is reflexive.[2] This will be shown below as a consequence of a general theorem on *pseudo-resolvents*.[3] A pseudo-resolvent $J(\lambda)$ is a function on a subset D of the complex plane to $E(X)$ satisfying the *resolvent equation*

$$J(\lambda)-J(\mu)=-(\lambda-\mu)J(\lambda)J(\mu),\quad \lambda,\mu\in D. \tag{1}$$

It follows directly from (1) that all $J(\lambda)$, $\lambda\in D$, have a common null space N and a common range R, which will be called respectively the null space and the range of the pseudo-resolvent under consideration. N is a closed subspace of X, but R need not be closed; we denote by $[R]$ the closure of R. Note that $J(\lambda)$ is a resolvent (of a closed linear operator A) if and only if $N=\{0\}$; in this case R coincides with the domain of A.

Theorem. *Let $J(\lambda)$, $\lambda\in D$, be a pseudo-resolvent with the null space N and the range R. Let there be a sequence $\{\lambda_n\}$, $n=1,2,\cdots$, such that*

$$\lambda_n\in D,\quad |\lambda_n|\to+\infty,\quad ||\lambda_n J(\lambda_n)||\le M=\text{const.} \tag{2}$$

Then we have

$$N\cap[R]=\{0\}. \tag{3}$$

If, in particular, X is l.s.w.c., then

$$X=N\oplus[R]. \tag{4}$$

1) See E. Hille and R. S. Phillips: Functional analysis and semi-groups, Am. Math. Soc. Colloq. Publ., Vol. 31, Theorems 12.3.1, 12.3.2, 12.4.1 and 12.5.1.

2) When X is a Hilbert space, this fact was noted by C. Foiaş, Bull. Soc. Math. France, **85**, 263 (1957).

3) Hille and Phillips: Footnote 1), pp. 126 and 183.

Corollary 1. *If* $J(\lambda)$ *is a pseudo-resolvent satisfying* (2) *and if* R *is dense in* X, *then* $J(\lambda)$ *is a resolvent.*[4)]

Corollary 2. *If* X *is l.s.w.c. and* $J(\lambda)=(\lambda I-A)^{-1}$ *is the resolvent of a closed linear operator* A *satisfying* (2), *then* A *is densely defined.*

Proof of the theorem. (2) implies that $||J(\lambda_n)||\to 0$ for $n\to\infty$. Setting $\lambda=\lambda_n$, $\mu=\lambda_1$ in (1) and making $n\to\infty$, we thus obtain

(5) $$||[\lambda_n J(\lambda_n)-I]J(\mu)||\to 0,\quad n\to\infty,\quad \mu=\lambda_1.$$

Since each $x\in R$ has the form $x=J(\mu)y$, it follows that

(6) $$\lambda_n J(\lambda_n)x\to x \qquad \text{for } x\in R.$$

Since $\{\lambda_n J(\lambda_n)\}$ is uniformly bounded, (6) can be extended to all $x\in[R]$. If $x\in N\cap[R]$, we have (6) and $J(\lambda_n)x=0$ so that $x=0$; this proves (3).

Assume now that X is l.s.w.c. For any $x\in X$, the sequence $\{\lambda_n J(\lambda_n)x\}$ is bounded; hence it contains a subsequence converging weakly to a $y\in X$. We may assume without loss of generality that

(7) $$\lambda_n J(\lambda_n)x\to y \qquad \text{weakly.}$$

This implies that $y\in[R]$ because R is weakly closed. On the other hand, the application of $J(\mu)$ to (7) gives $\lambda_n J(\lambda_n)J(\mu)x=J(\mu)\lambda_n J(\lambda_n)x \to J(\mu)y$ weakly. In view of (5), this gives $J(\mu)x=J(\mu)y$, $x-y\in N$. Thus we have $x\in N+[R]$ for any x. Combined with (3), this proves (4).

Remark. (4) and Corollary 2 may be false if X is not l.s.w.c., as is seen from the following example. Let $X=C[0,1]$ and $A=d^2/dx^2$ with the boundary conditions $f(0)=f(1)=0$. The resolvent of A satisfies the inequality $||(\lambda I-A)^{-1}||\leq 1/\mathrm{Re}\,\lambda$ for $\mathrm{Re}\,\lambda>0$, the resolvent set containing the half plane $\mathrm{Re}\,\lambda>0$. But A is *not* densely defined.

4) This may be used to justify the proof of Theorem 5.1 of H. F. Trotter, Pacific J. Math., **8**, 887 (1958), in which it is tacitly assumed that $N=\{0\}$.