107. *An Approximation Problem in Quasi-normed Spaces*

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(Comm. by K. KUNUGI, M.J.A., Oct. 12, 1959)

Recently a linear metric space with a quasi-norm was considered by M. Pavel, S. Rolewicz and Konda. In this Note, we shall consider an approximation problem [1, p. 79] in quasi-normed spaces.

A quasi-normed space E of order s $(0<s\leq 1)$ is a vector space over the real numbers on which is defined a non-negative real valued function $||x||$ called the *quasi-norm* such that

1) $||x||=0$ implies $x=0$,
2) $||x+y||\leq ||x||+||y||$,
3) $||\lambda x||=|\lambda|^s||x||$,

where s is independent to x of E.

Let $x_1, x_2,\cdots, x_n$ be a set of linearly independent elements of E. We shall consider $\varphi(\lambda_1, \lambda_2,\cdots, \lambda_n)=||y-\lambda_1 x_1-\lambda_2 x_2-\cdots-\lambda_n x_n||$.
For $\lambda_1, \lambda_2,\cdots, \lambda_n$; $\mu_1, \mu_2,\cdots, \mu_n$, we have

$$\begin{aligned}&|\varphi(\lambda_1, \lambda_2,\cdots, \lambda_n)-\varphi(\mu_1, \mu_2,\cdots, \mu_3)|\\ &=\left|\left|\left|y-\sum_{i=1}^n \lambda_i x_n\right|\right|-\left|\left|y-\sum_{i=1}^n \mu_i y_i\right|\right|\right|\\ &\leq\left|\left|\sum_{i=1}^n(\lambda_i-\mu_i)x\right|\right|\leq\sum_{i=1}^n\left|\left|(\lambda_i-\mu_i)x_i\right|\right|\\ &=\sum_{i=1}^n|\lambda_i-\mu_i|^s||x_i||.\end{aligned}$$

This shows that $\varphi(\lambda_1, \lambda_2,\cdots, \lambda_n)$ is a continuous function of $\lambda_1, \lambda_2,\cdots, \lambda_n$. The function

$$\psi(\lambda_1, \lambda_2,\cdots, \lambda_n)=||\lambda_1 x_1+\lambda_2 x_2+\cdots+\lambda_n x_n||$$

on the unit sphere $\sum_{i=1}^n \lambda_i^2=1$ in n-dimensional space takes minimal value α, since the unit sphere is compact. The minimal value α is positive, since $x_1, x_2,\cdots, x_n$ are linearly independent.

Let M be a given positive number, and let

$$\left(\sum_{i=1}^n \lambda_i^2\right)^{s/2}>\frac{1}{\alpha}(M+||y||),$$

then we have

$$\begin{aligned}\varphi(\lambda_1, \lambda_2,\cdots, \lambda_n)&\geq\left|\left|\sum_{i=1}^n \lambda_i x_i\right|\right|-||y||\\ &=(\sum\lambda_i^2)^{s/2}\left|\left|\sum_{i=1}^n\frac{\lambda_i}{\sqrt{\sum_{k=1}^n \lambda_k^2}}x_i\right|\right|-||y||\\ &\geq\left(\sum_{i=1}^n \lambda_i^2\right)^{s/2}\alpha-||y||>M.\end{aligned}$$

Therefore, for every $\lambda_1, \lambda_2, \cdots, \lambda_n$ such that $\sum_{i=1}^{n} \lambda_i^2 \geq r^2$ for some positive r, we have $\varphi(\lambda_1, \lambda_2, \cdots, \lambda_n) > \| y \|$. On the sphere $\sum_{i=1}^{n} \lambda_i^2 \leq r^2$, the function $\varphi(\lambda_1, \lambda_2, \cdots, \lambda_n)$ takes the minimal value γ. From $\varphi(0, 0, \cdots, 0) = \| y \|$, we have $\gamma \leq \| y \|$. Hence $\varphi(\lambda_1, \lambda_2, \cdots, \lambda_n)$ takes the minimal value λ on the n-dimensional space, and we have the following approximation

Theorem. *The function*

$$\varphi(\lambda_1, \lambda_2, \cdots, \lambda_n) = \| y - \lambda_1 x_1 - \cdots - \lambda_n x_n \|$$

on a quasi-normed space takes the minimal value for some $\lambda_1, \lambda_2, \cdots, \lambda_n$.

Reference

[1] L. A. Lujsternik und W. L. Sobolew: Elemente der Funktionalanalysis, Berlin (1955).