104. On Singular Perturbation of Linear Partial Differential Equations with Constant Coefficients. I

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1. Introduction. Let $(t, x)=(t, x_1,\cdots,x_m)$ be $m+1$ real variables in $t\geqq 0$, $x\in E^m$, where E^m denotes the m-dimensional Euclidean space. Let L_ε be an $r\times r$ matrix of differential operators with constant coefficients depending on a parameter ε

$$L_\varepsilon=\sum_{j=1}^{l} P_j(\partial_x, \varepsilon)\,\partial_t^j\,{}^{1)}$$

where $P_j(\xi, \varepsilon)$ are $r\times r$ matrices of polynomials in $\xi=(\xi_1,\cdots,\xi_m)$, whose coefficients depends on $\varepsilon\geqq 0$ *continuously*, and let us consider a system of partial differential equations

$$(1)\qquad L_\varepsilon[u]=f(t, x, \varepsilon),$$

where $u=(u_\rho\ \rho\downarrow 1,\cdots,r)$ $f=(f_\rho\ \rho\downarrow 1,\cdots,r)$.[2] We assume that $P_l(\xi,\varepsilon)=P_l(\varepsilon)$ does not contain ξ and

$$(2)\qquad \det\,(P_l(\varepsilon))\neq 0 \text{ for } \varepsilon>0.$$

In this note we are concerned with showing the relationship of (1), as $\varepsilon\downarrow 0$, to a particular solution of a related system (for $\varepsilon=0$)

$$(1^\circ)\qquad L_0[u]=f(t, x, 0),$$

especially when L_0 is *degenerated*, i.e.

$$(2^\circ)\qquad \det\,(P_l(0))=0.\,{}^{3)}$$

Let C_0^∞ be the set of all on E^m infinite times continuously differentiable complex valued functions with compact carrier. For any $u\in C_0^\infty$ we define the norm $\|u\|_p$ by

$$(3)\qquad \|u\|_p^2=\int_{E^m}\sum_{|\nu|\leq p}|\partial_1^{\nu_1}\cdots\partial_m^{\nu_m}u(x)|^2\,dx,{}^{4)}\quad (|\nu|=\nu_1+\cdots+\nu_m).$$

The completion of C_0^∞ with respect to the norm (3) will be denoted by H_p. H_p is a kind of Hilbert space. One sees easily

$$H_p\supset H_{p'} \text{ and } \|u\|_p\leqq\|u\|_{p'} \text{ if } p<p'.$$

We set $H_\infty=\bigcap_{p<\infty}H_p$, then H_∞ is a linear topological space with a sequence of semi-norms $\|u\|_p$ $(p=0, 1, 2,\cdots)$ for $u\in H_\infty$. H_∞ is dense in H_p for any p, and C_0^∞ is dense in H_∞ (hence in H_p).

Let $\hat{\varphi}$ be the Fourier transform of $\varphi\in H_p$,

$$(4)\qquad \hat{\varphi}(\xi)=\frac{1}{\sqrt{2\pi}^{\,m}}\int_{E^m}e^{-i\xi\cdot x}\varphi(x)dx=\mathfrak{F}[\varphi],$$

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1) We use ∂_t for ∂/∂_t, and ∂_x for $\partial/\partial x_1,\cdots,\partial/\partial x_m$.

2) $(u_\rho\ \rho\downarrow 1,\cdots,r)$ means the r-dimensional vector (column) with the components $(u_1,\cdots,u_r)$.

3) The condition (2) is not essential in the general consideration.

4) ∂_μ is the abbreviation of $\partial/\partial x_\mu$.

then $\varphi \in H_p$ is equivalent to $(1+|\xi|^2)^{p/2}\hat{\varphi}(\xi)\in L^2$ and

$$(5) \qquad \|\varphi\|_p^2 = \int_{E^m} (1+|\xi|^2)^p |\hat{\varphi}(\xi)|^2 d\xi = \|\hat{\varphi}\|_p'^2.$$

The complete space of all measurable complex valued functions $\hat{\varphi}$ such that $\|\hat{\varphi}\|_p' < \infty$ will be denoted by $\hat{H}_p$.[5] The Fourier transform $\mathfrak{F}$ is a unitary transformation of H_p onto $\hat{H}_p$.

For any real number $p \gtrless 0$, we can define the norm $\|\varphi\|_p$ for $\varphi \in C_0^\infty$ by (5). If $p \geqq 0$, then the completion of C_0^∞ which we denote by H_p, with respect to the norm (5) is the set of all complex valued measurable functions such that $\|\varphi\|_p < \infty$.[5] But if $p < 0$, the completion of C_0^∞ with respect to (5) consists from a class of distributions by *L. Schwarz*. The Fourier transform of H_p, denoted by $\hat{H}_p$, even if $p < 0$, is the set of all measurable functions $\hat{\varphi}$[5] such that $\|\hat{\varphi}\|_p' < \infty$ by (5).

Let $D^{(k)}$ be any differential operator with constant coefficients of order k, then $D^{(k)}$ is a bounded linear mapping of H_p into H_{p-k}.

Let F_x be any linear functional space, whose elements are functions of $x \in E^m$, and $\varphi(t)$ be a variable element of F_x depending on a real parameter t in an interval J. We say "$\varphi(t)$ is F_x-continuous in $t \in J$" if the mapping $t \in J \longrightarrow \varphi(t) \in F_x$ is continuous, and "$\varphi(t)$ is F_x-differentiable at $t=t_0$" if

$$(6) \qquad (t-t_0)^{-1}\{\varphi(t)-\varphi(t_0)\} \to \varphi'(t_0) \text{ in } F_x \text{ as } t \to t_0.$$

We use the notation $\varphi'(t) = \frac{d}{dt}\varphi(t)$, if $\varphi'(t)$ defined by (6) has meaning for t in an interval. If $D^{(k)}$ is a differential operator in $x \in E^m$ with constant coefficients of order k and $\varphi(t)$ is $H_{p,x}$-continuous in t, then $D^{(k)}\varphi(t)$ is $H_{p-k,x}$-continuous, and if $\varphi(t)$ is $H_{p,x}$-differentiable in t then $D^{(k)}\varphi(t)$ is $H_{p-k,x}$-differentiable in t and

$$\frac{d}{dt}\left\{D^{(k)}\varphi(t)\right\} = D^{(k)}\left\{\frac{d}{dt}\varphi(t)\right\}.$$

Let $u=u(t)=u(t,x)$ be l times continuously $H_{p,x}$-differentiable in $t \in J$, and L be a differential operator in (t,x) with constant coefficients defined by

$$(7 \qquad L[u] = \sum_{j=0}^{l} P_j(\partial_x)\,\partial_t^j u(t,x),$$

where $P_j(\xi)$ are polynomials in $\xi=(\xi_1,\cdots,\xi_m)$ of degree at most k with constant coefficients. Then $L[u](t)$ is $H_{p-k,x}$-continuous in $t \in J$. Putting

$$(8) \qquad L[u](t) = f(t)$$

we say $u(t)$ is an H_p-*solution* of the equation (8).

Now we extend the operator L as follows:

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5) Strictly speaking, each element of the space is a such class of functions, that any pair of which differ at most on a set of measure zero.

Definition 1. *Let* $\{u_\nu(t)\}_{\nu=1}^{\infty}(u_\nu(t)=u_\nu(t,x))$ *be a sequence of* l *times continuously* $H_{p,x}$*-differentiable functions in* $t\in J$, *such that as* $\nu\to\infty$, $u_\nu(t)\to u(t)$ *in* $H_{p,x}$ *quasi-uniformly for* $t\in J$,[6] *and* $L[u_\nu(t)]\to f(t)$ *in* $H_{p-k,x}$ *quasi-uniformly for* $t\in J$. *Then we define* $L[u(t)]=f(t)$ *for* $t\in J$, *and we say* $u(t)$ *is a* **generalized H_p-solution** *of* (8).

A generalized H_p-solution is naturally $H_{p,x}$-continuous in t, but it is not necessarily $H_{p,x}$-differentiable in t. This extension of the operator L is free from absurdity. Because, L is a pre-closed linear operator as follows:

If $u_\nu(t)\to 0$ in $H_{p,x}$ quasi-uniformly for $t\in J$, and $L[u_\nu(t)]\to f(t)$ in $H_{p-k,x}$ quasi-uniformly for $t\in J$, then $f(t)=0$ for $t\in J$.

We say "a system $u_1(t),\cdots,u_r(t)$ has property (P)" if each $u_\rho(t)$ $(\rho=1,\cdots,r)$ has the property (P). The above definitions and related statements can be all extended to a system of functions and system of operators in a quite similar way, so that we need not explain them in detail.

2. Preliminary theorems. In the following let us give some preliminary theorems without proof.

Let L be a matrix of differential operators

$$L=\sum_{j=1}^{l}P_j(\partial_x)\partial_t^j$$

where $P_j(\xi)$ are $r\times r$ matrices of polynomials in $\xi=(\xi_1,\cdots,\xi_m)$ at most of order k with constant coefficients, and $P_l(\xi)=P_l$ be a constant matrix such that $\det(P_l)\neq 0$.

Theorem 1. *If* $u=u(t)=u(t,x)$ *is a generalized* H_p*-solution of* $L[u]=f(t)$ *for* $t\in J$, *then there exists a sequence of* l *times continuously* $C_{0,x}^{\infty}$*-differentiable* $u_\nu(t)=u_\nu(t,x)$ *for* $t\in J$, *such that as* $\nu\to\infty$, $u_\nu(t)\to u(t)$ *in* $H_{p,x}$ *quasi-uniformly for* $t\in J$, *and* $L[u_\nu(t)]\to f(t)$ *in* $H_{p-k,x}$ *quasi-uniformly for* $t\in J$.

We associate the partial differential equation $L[u]=f(t)$ with the following ordinary differential equation

$$\sum_{\mu=0}^{l}P_\mu(i\xi)\left(\frac{d}{dt}\right)^{\mu}Y=0. \tag{2.1}$$

Let $Y_j(t,\xi)$ be matricial solutions of (2.1) with the initial conditions

$$(\partial_t^{k-1}Y)_{t=0}=\delta_{jk}\,1.$$

Theorem 2. *If there exist constants* C *and* q *such that*

$$|Y_j(t,\xi)|^{7)}\leqq C\sqrt{1+|\xi|^2}^{\,q}\quad (j=1,\cdots,l)\ \textit{for}\ 0\leqq t\leqq T \tag{2.2}$$

and $f(t,x)$ *is* $H_{p,x}$*-continuous in* $0\leqq t\leqq T$, *then the partial differential equation*

$$L[u]=\sum_{\mu=0}^{l}P_\mu(\partial_x)\partial_t^{\mu}u=f(t,x) \tag{2.3}$$

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6) "Quasi-uniform for $t\in J$" means "uniform for any compact part of J".

7) $|Y|$ is the norm of the matrix Y, defined by $|Y|=\operatorname{Sup}_{u\neq 0}\{|Yu|/|u|\}$.

has generalized H_{p-q}-solution $u=u(t,x)$ with the initial conditions

$$\partial_t^{j-1}u(0,x)=\varphi_j(x) \qquad (j=1,\cdots,l),$$

where φ_j are arbitrary functions of $H_{p,x}$. Further if $p\geqq q$, then this solution $u=u(t,x)$ is represented by

$$(2.4)\qquad \begin{aligned} u(t,x)=&\sum_{j=1}^{l}\frac{1}{\sqrt{2\pi}^m}\int_{E^m} e^{ix\cdot\xi}Y_j(t,\xi)\hat{\varphi}_j(\xi)d\xi \\ &+\frac{1}{\sqrt{2\pi}^m}\int_{E^m} e^{ix\cdot\xi}d\xi\int_0^t P_l^{-1}Y_l(t-\tau,\xi)\ \hat{f}(\tau,\xi)dt, \end{aligned}$$

where $\hat{\varphi}_j$ and $\hat{f}$ are Fourier transforms of φ_j and f respectively as functions of x.

Further if

$$|\partial_t^{k-1}Y_j(t,\xi)|<C\sqrt{1+|\xi|^2}^{\,q}\ \textit{for}\ 0\leqq t\leqq T,$$

$k=1,\cdots,l$, $j=1,\cdots,l$, then the solution $u=u(t,x)$ is an H_{p-q}-solution in proper sense.

3. **Stability.** Consider a system of equations containing a parameter ε

$$(3.1)\qquad L_\varepsilon[u]=\sum_{\mu=0}^{l}P_\mu(\partial_x,\varepsilon)\partial_t^\mu u=f_\varepsilon(t,x),$$

where $P_\mu(\xi,\varepsilon)$ are $r\times r$ matrices of polynomials in $\xi=(\xi_1,\cdots,\xi_m)$ with constant coefficients depending on ε continuously for $\varepsilon\geqq 0$, and $P_l(\varepsilon)=P_l(\xi,\varepsilon)$ depends on ε only and

$$\det(P_l(\varepsilon))\neq 0\ \text{for}\ \varepsilon>0.$$

Definition 2. *We say that the equation* (3.1) *is* **H_p-stable** *for $\varepsilon\downarrow 0$ in $0\leqq t\leqq T$ with respect to a particular solution $u=u_0(t)$ of* (3.1) *for $\varepsilon=0$, if and only if,*

$$u_\varepsilon(t)\to u_0(t)\ \textit{in}\ H_{p,x}\ \textit{uniformly for}\ 0\leqq t\leqq T,$$

whenever

$$(3.2)\qquad f_\varepsilon(t)=f_\varepsilon(t,x)\to f_0(t)\ \textit{in}\ H_{p,x}\ \textit{uniformly for}\ 0\leqq t\leqq T,$$

and $u_\varepsilon(t)=u(t,x,\varepsilon)$ is a generalized H_p-solution of (3.1) *such that*

$$(3.3)\qquad \partial_t^{j-1}u_\varepsilon(0)\to\partial_t^{j-1}u_0(0)\ \textit{in}\ H_{p,x}\ (j=1,\cdots,l).$$

Theorem 3. *Let degree of $\{P_\mu(\xi,\varepsilon)-P_\mu(\xi,0)\}=k$ $(\mu=0,\cdots,l)$, and let $u=u_0(t)$ be an l times continuously $H_{p+k,x}$-differentiable solution of* (3.1) *for $\varepsilon=0$ in $0\leqq t\leqq T$. In order that* (3.1) *be H_p-stable for $\varepsilon\downarrow 0$ with respect to $u=u_0(t)$ in $0\leqq t\leqq T$, it is necessary and sufficient that, there exist constants $\varepsilon_0>0$ and C such that*

$$(3.4)\qquad \operatorname*{Sup}_{\xi\in E^m}|Y_j(t,\xi,\varepsilon)|\leqq C\ \textit{for}\ 0\leqq t\leqq T,\ 0<\varepsilon\leqq\varepsilon_0,$$

and

$$(3.5)\qquad \operatorname*{Sup}_{\xi\in E^m}\int_0^T|P_l(\varepsilon)^{-1}Y_l(t,\xi,\varepsilon)|dt\leqq C\ \textit{for}\ 0<\varepsilon\leqq\varepsilon_0,$$

where $y=Y_j(t,\xi,\varepsilon)$ are matricial solutions of

$$(3.6)\qquad \sum_{\mu=0}^{l}P_\mu(i\xi,\varepsilon)\left(\frac{d}{dt}\right)^\mu y=0$$

with the initial conditions $\partial_t^{k-1}Y_j(0,\xi,\varepsilon)=\delta_{kj}1$ $(k=1,\cdots,l)$.

Proof. Necessity of (3.4): Let $v=v_\varepsilon(t)$ be the solution of

$$L_\varepsilon[v]=0 \tag{3.7}$$

with the initial conditions $\partial_t^{j-1}v_\varepsilon(0)=\partial_t^{j-1}u_\varepsilon(0)-\partial_t^{j-1}u_0(0)$ $(j=1,\cdots,l)$. One sees easily, it is necessary that

$$v_\varepsilon(t)\to 0 \text{ in } H_{p,x} \text{ uniformly for } 0\leqq t\leqq T. \tag{3.8}$$

Now assume that for any $\varepsilon_0>0$, there did not exist such C that (3.4) holds. Then, for a certain j, there are sequences $\{\varepsilon_\nu\}$ and $\{t_\nu\}$ such that, $\varepsilon_\nu\downarrow 0$ as $\nu\to\infty$, $0\leqq t_\nu\leqq T$, and a sequence of spheres $\{S_\nu\}$, $S_\nu=\{\xi;\ |\xi-\xi^{(\nu)}|<\delta_\nu\}$, such that

$$\begin{aligned} &|Y_j(t_\nu,\xi,\varepsilon_\nu)|>\nu \qquad \text{for } \xi\in S_\nu,\\ &2^{-1}<\sqrt{1+|\xi|^2}^{\,p}\Big/\sqrt{1+|\xi^{(\nu)}|^2}^{\,p}<2 \text{ for } \xi\in S_\nu. \end{aligned} \tag{3.9}$$

We set

$$v_\nu(t,x)=\frac{\alpha_\nu}{\sqrt{2\pi}^{\,m}}\int_{S_\nu} e^{ix\cdot\xi}Y_j(t,\xi,\varepsilon_\nu)d\xi,$$

with $\alpha_\nu=(\text{measure of } S_\nu)^{-1}\sqrt{1+|\xi^{(\nu)}|^2}^{\,-p}$. Then $v=v_\nu(t,x)$ is an H_∞-solution of (3.7) such that, by (3.9),

$$\|\partial_t^{j-1}v_\nu(0)\|_p\leqq 2\nu^{-1}\to 0,\quad \partial_t^{k-1}v_\nu(0)=0 \text{ for } k\neq j,$$

and $\|v_\nu(t_\nu)\|_p\geqq 1/2$. This contradicts with (3.8). The condition (3.4) is thus necessary.

Necessity of (3.5): If (3.5) did not hold for any $\varepsilon_0>0$ and C, then there would exist a sequence $\{\varepsilon_\nu\}$, $\varepsilon_\nu\downarrow 0$ and a sequence of spheres $\{S_\nu\}\subset E^m$ such that

$$\int_0^T|P_l(\varepsilon_\nu)^{-1}Y_l(T-\tau,\xi,\varepsilon_\nu)|\,d\tau>\nu \quad \text{for } \xi\in S_\nu. \tag{3.10}$$

Let $u=u_\nu(t)=u_\nu(t,x)$ be generalized H_p-solution of (3.1) with the initial conditions $\partial_t^{j-1}u_\nu(0)=\partial_t^{j-1}u_0(0)$ $(j=1,\cdots,l)$. Then $v=v_\nu(t)=u_\nu(t)-u_0(t)$ is a generalized H_p-solution of

$$L_{\varepsilon_\nu}[u]=g_\nu(t) \tag{3.11}$$

with $g_\nu(t)=g_\nu(t,x)=\{L_{\varepsilon_\nu}-L_0\}[u_0]+f_{\varepsilon_\nu}(t)-f_0(t)$, with the initial conditions $\partial_t^{j-1}v_\nu(0)=0$ $(j=1,\cdots,l)$. By Theorem 2, since $g_\nu(t)$ is $H_{p,x}$-continuous and (3.4) holds,

$$v_\nu(t,x)=\frac{1}{\sqrt{2\pi}^{\,m}}\int_{E^m} e^{ix\cdot\xi}\left\{\int_0^t P_l(\varepsilon)^{-1}Y_l(t-\tau,\xi,\varepsilon_\nu)\hat{g}_\nu(\tau,\xi)d\tau\right\}d\xi, \tag{3.12}$$

where $\hat{g}_\nu(t,\xi)$ denotes the Fourier transform of $g_\nu(t,x)$ as the function of x. Since $\{L_{\varepsilon_\nu}-L_0\}[u_0]$ is $H_{p,x}$-continuous and

$$\{L_{\varepsilon_\nu}-L_0\}[u_0]\to 0 \text{ in } H_{p,x} \text{ uniformly for } 0\leqq t\leqq T,$$

$g_\nu(t)$ may be any $H_{p,x}$-continuous function such that $g_\nu(t)\to 0$ in $H_{p,x}$ uniformly for $0\leqq t\leqq T$.

Now by (3.10) we can find a continuous function $\psi_\nu(t,\xi)$ in $0\leqq t\leqq T$, $\xi\in S_\nu$, such that

$$(3.13)\quad \begin{cases} |\psi_\nu(t,\xi)|\leqq 1, \\ \left|\int_0^T P_l(\varepsilon_\nu)^{-1}Y_l(T-\tau,\xi,\varepsilon_\nu)\psi_\nu(\tau,\xi)\,d\tau\right|>\nu \text{ for } \xi\in S_\nu. \end{cases}$$

We set

$$(3.14)\qquad g_\nu(t,x)=\frac{\nu^{-1}|S_\nu|^{-1/2}}{\sqrt{2\pi}^m}\int_{S_\nu} e^{ix\cdot\xi}\psi_\nu(t,\xi)\sqrt{1+|\xi|^2}^{-p}d\xi,$$ [8]

hence

$$\hat{g}_\nu(t,\xi)=\begin{cases} \nu^{-1}|S_\nu|^{-1/2}\sqrt{1+|\xi|^2}^{-p}\psi_\nu(t,\xi) & \text{for } \xi\in S_\nu, \\ 0 & \text{for } \xi\bar{\in} S_\nu. \end{cases}$$

Then $\|g_\nu(t,x)\|_p\leqq\nu^{-1}\to 0$. But by (3.12), (3.13) and (3.14)

$$\|v_\nu(T)\|_p\geqq 1.$$

This contradicts with $v_\nu(t)=u_{\varepsilon_\nu}(t)-u_0(t)\to 0$ in $H_{p,x}$ uniformly for $0\leqq t\leqq T$. The condition (3.5) is thus necessary.

Sufficiency of the conditions. Put $v_\varepsilon(t)=u_\varepsilon(t)-u_0(t)$, then $v_\varepsilon(t)$ is given by

$$(3.15)\quad \begin{aligned} v_\varepsilon(t,x)=&\sum_{j=1}^{l}\frac{1}{\sqrt{2\pi}^m}\int_{E^m} e^{ix\cdot\xi}Y_j(t,\xi,\varepsilon)\partial_t^{j-1}\{\hat{u}_\varepsilon(0,\xi)-\hat{u}_0(0,\xi)\}d\xi \\ &+\frac{1}{\sqrt{2\pi}^m}\int_{E^m} e^{ix\cdot\xi}\left\{\int_0^t P_l(\varepsilon)^{-1}Y_l(t-\tau,\xi,\varepsilon)\hat{g}_\varepsilon(\tau,\xi)d\tau\right\}d\xi, \end{aligned}$$

where $g_\varepsilon(t,x)=L_\varepsilon[u_0]-L_0[u_0]+f_\varepsilon(t)-f_0(t)$ and $\hat{g}_\varepsilon(t,\xi)=\mathfrak{F}_x[g_\varepsilon(t,x)](\xi)$. From (3.4), (3.5) and (3.15) we can easily derive

$$\|v_\varepsilon(t,x)\|_p\to 0 \text{ uniformly for } 0\leqq t\leqq T,$$

if $\|\partial_t^{j-1}\{u_\varepsilon(0)-u_0(0)\}\|_p\to 0$ and $\|f_\varepsilon(t)-f_0(t)\|_p\to 0$ uniformly for $0\leqq t\leqq T$. Q.E.D.

8) $|S_\nu|$ denotes the measure of S_ν.