103. A Characteristic Property of L_p-Spaces ($p>1$)

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In the theory of L_p-spaces, $p>1$, the fundamental rôle is played by Hölder's inequality:

$$(1)\qquad \int_0^1 f(t)\,g(t)\,dt \leqq \Big(\int_0^1 |f(t)|^p\,dt\Big)^{1/p}\Big(\int_0^1 |g(t)|^q\,dt\Big)^{1/q}$$

where $f(t)\in L_p$, $g(t)\in L_q$ and $q=p/p-1$.

This inequality is usually proved by making use of the following special Young's inequality:

$$(2)\qquad \int_0^1 f(t)\,g(t)\,dt \leqq \frac{1}{p}\int_0^1 f(t)|^p\,dt+\frac{1}{q}\int_0^1 |g(t)|^q\,dt.$$

It is well known that, for the function

$$(3)\qquad g(t)=|f(t)|^{p-1}\,sgn\,f(t)=Tf(t),$$

we get the equality sign in (1). Namely, if the equality holds in (2) for a pair of functions, then for the same pair the equality holds in (1). The purpose of this paper is to show that this property is characteristic for L_p-spaces, $p>1$.

The transformation T in (3) has the following properties:

(i) $x\geqq y\geqq 0$ implies $Tx\geqq Ty\geqq 0$;

(ii) $(Tx)[y]=T([y]x)$ for any projector $[y]$;[1]

(iii) $T(-x)=-Tx$.

A transformation T from a universally continuous semi-ordered linear space R into its conjugate space $\overline{R}$;[2] with the above conditions (i)–(iii) is said to be *conjugately similar*.

A function $\|x\|$ on a universally continuous semi-ordered linear space is called a norm if

(i) $\|x\|\geqq 0$; $\|x\|=0$ implies $x=0$;

(ii) $\|\alpha x\|=|\alpha|\,\|x\|$;

(iii) $\|x+y\|\leqq\|x\|+\|y\|$;

(iv) $x\geqq y\geqq 0$ implies $\|x\|\geqq\|y\|\geqq 0$.

The conjugate norm is defined by

$$\|\overline{x}\|=\sup_{\|x\|\leqq 1}(\overline{x},x)\quad (x\in R,\ \overline{x}\in\overline{R}).$$

We prove the following

Theorem. Let R be a normed universally continuous semi-ordered

1) $[y]x=\bigcup_{n=1}^{\infty}(x\frown n|y|)$ if $x\geqq 0$ and $[y]x=[y]x^{+}-[y]x^{-}$ for any $x\in R$.

2) The conjugate space of a normed semi-ordered linear space is the set of norm-bounded and universally continuous linear functionals. See [1, §31].

linear space which has at least two linearly independent elements and its conjugate norm be strictly convex. If there exists a one-to-one conjugately similar correspondence T with the following condition

$$(4) \qquad (Tx, x)=\| Tx \| \cdot \| x \| \quad (0 \leqq x \in R),$$

then we can find a number $p>1$ such that

$$T\xi x=\xi^{p-1}\, Tx$$

for any number $\xi>0$ and $x \in R$.

In the proof, we make use of the fact that the existence of such T enables us to define on R a modular $m(x)$ which satisfies the following conditions:

(i) $0<m(x)<+\infty$ for every $0 \neq x \in R$;

(ii) $m(\xi x)$ is a convex function of $\xi>0$;

(iii) $m(x+y)=m(x)+m(y)$ if x and y are mutually orthogonal;

(iv) $x \geqq y \geqq 0$ implies $m(x) \geqq m(y)$;

(v) $0 \leqq x_\lambda \uparrow x$ implies $m(x)=\sup_{\lambda \in \Lambda} m(x_\lambda)$.

In fact, the modular is defined by

$$m(x)=\int_0^1 (T\xi x, x)\, d\xi$$

when x is non-negative and

$$m(x)=m(x^+)-m(x^-)$$

for any $x \in R$.

Conversely, if m is once defined, T is characterized by the following equation:

$$(Tx, x)=m(x)+\overline{m}(\overline{x})^{3)}$$

which is a generalization of (3). This is the reason why we assert that *our theorem gives a characterization of L_p by means of the relation between Young's and Hölder's inequalities.*[4]

Proof of Theorem. It follows from (4) that

$$(T\xi x, x)=\| T\xi x \| \cdot \| x \|$$

for any $\xi>0$. Therefore, strict convexity of the conjugate norm implies the existence of such a function $f_x(\xi)$ that

$$(5) \qquad T\xi x=f_x(\xi)\, Tx \quad (\xi>0,\ 0 \leqq x \in R).$$

Putting

$$m(x)=\int_0^1 (T\xi x, x)\, d\xi,$$

we get by (5) that

$$m(\xi[p]x)=\int_0^\xi (T\eta[p]x, x)\, d\eta$$

$$=\int_0^\xi (T\eta x, [p]x)\, d\eta$$

3) $\overline{m}(\overline{x})=\sup_{x \in R} \{(\overline{x}, x)-m(x)\}$.

4) In this sense, our theorem is closely related to §3 of [2].

$$=\int_0^\xi f_x(\eta)\,d\eta\cdot(T[p]x,x).$$

Hence it follows that

$$(6)\qquad \frac{m(\xi[p]x)}{m([p]x)}=\frac{\int_0^\xi f_x(\eta)\,d\eta}{\int_0^1 f_x(\eta)\,d\eta}=\frac{m(\xi x)}{m(x)}$$

for any $\xi>0$ and $[p]$ with $[p]x\neq 0$.

Now, we will prove that, if (6) holds for any element x, we can find a number $p>1$ such that

$$m(\xi x)=\xi^p m(x) \qquad (\xi>0).$$

To prove this, take a positive element x. Since R is at least two dimensional, there exists $y>0$ such that $x\frown y=0$. Then, putting $z_\xi=\xi x+y$, we have by (6) that

$$\frac{m(\eta z_\xi)}{m(z_\xi)}=\frac{m(\eta[x]z_\xi)}{m([x]z_\xi)}=\frac{m(\xi\eta x)}{m(\xi x)} \qquad (\xi,\eta>0)$$

and

$$\frac{m(\eta z_\xi)}{m(z_\xi)}=\frac{m(\eta[y]z_\xi)}{m([y]z_\xi)}=\frac{m(\eta y)}{m(y)} \qquad (\xi,\eta>0).$$

Therefore,

$$\frac{m(\xi\eta x)}{m(x)}=\frac{m(\eta x)}{m(x)}\cdot\frac{m(\xi x)}{m(x)} \qquad (\xi,\eta>0).$$

Since $m(\xi x)$ is continuous with respect to $\xi>0$, we can find $p\geqq 1$ such that

$$m(\xi x)=\xi^p m(x) \qquad (\xi>0).$$

Here, p must be strictly greater than one, because T is one-to-one.

From the definition of m, it follows that

$$(T\xi x,x)=\xi^{p-1}(Tx,x)$$

and therefore,

$$T\xi x=\xi^{p-1}Tx.$$

REMARK 1. The conclusion of this theorem means that R can be represented by a subset of L_p-space, $p>1$, on some measure space. If R is reflexive as a vector lattice, R is represented by an L_p-space.

REMARK 2. The case of one-dimensional space is exceptional. Let Φ and Ψ be Young's complementary functions, by which we can consider the set of all real numbers as an Orlicz space. For the norm $\|x\|_\Phi=\inf\{\xi^{-1}\colon \Phi(\xi x)\leqq 1,\ \xi\geqq 0\}$ and its conjugate norm $\|x\|_\Phi=\inf_\xi[1+\Psi(\xi x)]\cdot\xi^{-1}$, we always have $(Tx,x)=\|Tx\|_\Psi\cdot\|x\|_\Phi$, where Tx is defined by $\varphi(x)$, the left-hand derivative of $\Phi(x)$.

References

[1] H. Nakano: Modulared Semi-ordered Linear Spaces, Tokyo (1950).
[2] S. Yamamuro: On conjugate spaces of Nakano spaces, Trans. Amer. Math. Soc., **90**, 291–311 (1959).