102. On Compactness of Weak Topologies

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Let R be a space and $\mathfrak{a}_\lambda$ $(\lambda\in\Lambda)$ a system of mappings of R into topological spaces S_λ with neighbourhood systems $\mathfrak{N}_\lambda$ $(\lambda\in\Lambda)$. Concerning the weak topology of R by $\mathfrak{a}_\lambda$ $(\lambda\in\Lambda)$, i.e. the weakest topology of R for which all $\mathfrak{a}_\lambda$ $(\lambda\in\Lambda)$ are continuous, we have (H. Nakano: Topology and Linear Topological Spaces, Tokyo (1951), §19, Theorem 4. This book will be denoted by TLTS):

Theorem 1. *If all S_λ $(\lambda\in\Lambda)$ are compact Hausdorff spaces, then, in order that the weak topology of R be compact, it is necessary and sufficient that for any system of points $a_\lambda\in S_\lambda$ $(\lambda\in\Lambda)$ subject to the condition*

$$\text{(F)}\qquad \bigcap_{\nu=1}^{n}\mathfrak{a}_{\lambda_\nu}^{-1}(U_{\lambda_\nu})\neq\phi$$

for every finite number of open sets $a_{\lambda_\nu}\in U_{\lambda_\nu}\in\mathfrak{N}_{\lambda_\nu}$, $\lambda_\nu\in\Lambda$ $(\nu=1,2,\cdots,n)$, we can find a point $x\in R$ for which $\mathfrak{a}_\lambda(x)=a_\lambda$ for every $\lambda\in\Lambda$.

In the sequel, we consider generalization of this theorem in the case where S_λ $(\lambda\in\Lambda)$ are merely compact.

Theorem 2. *If all S_λ $(\lambda\in\Lambda)$ are compact and for any system of points $a_\lambda\in S_\lambda$ $(\lambda\in\Lambda)$ subject to the condition* (F), *we can find a point $x\in R$ for which $\mathfrak{a}_\lambda(x)\in\{a_\lambda\}^-$ for every $\lambda\in\Lambda$, then the weak topology of R is compact.*

Proof. Let $\mathfrak{K}$ be a maximal system of sets of R subject to the condition (I) $\bigcap_{\nu=1}^{n}K_\nu\neq\phi$ for every finite number of sets $K_\nu\in\mathfrak{K}$ $(\nu=1,2,\cdots,n)$. We see easily then that $A\frown K\neq\phi$ for all $K\in\mathfrak{K}$ implies $A\in\mathfrak{K}$, and L, $K\in\mathfrak{K}$ implies $L\frown K\in\mathfrak{K}$. For any $\lambda\in\Lambda$, we have obviously $\bigcap_{\nu=1}^{n}\mathfrak{a}_\lambda(K_\nu)\neq\phi$ for every finite number of sets $K_\nu\in\mathfrak{K}$ $(\nu=1,2,\cdots,n)$, and hence $\bigcap_{K\in\mathfrak{K}}\mathfrak{a}_\lambda(K)^-\neq\phi$, because S_λ is compact by assumption. For a point $a_\lambda\in\bigcap_{K\in\mathfrak{K}}a_\lambda(K)^-$, we have

$$\mathfrak{a}_\lambda^{-1}(U)\in\mathfrak{K}\quad\text{for } a_\lambda\in U\in\mathfrak{N}_\lambda,$$

because for $a_\lambda\in U\in\mathfrak{N}_\lambda$, $K\in\mathfrak{K}$ we have obviously

$$\mathfrak{a}_\lambda(K\frown\mathfrak{a}_\lambda^{-1}(U))=\mathfrak{a}_\lambda(K)\frown U\neq\phi$$

which yields $K\frown\mathfrak{a}_\lambda^{-1}(U)\neq\phi$. Therefore the system of points a_λ $(\lambda\in\Lambda)$ satisfies the condition (F), and hence we can find a point $x\in R$ by assumption such that $\mathfrak{a}_\lambda(x)\in\{a_\lambda\}^-$ for every $\lambda\in\Lambda$. For such a point $x\in R$, we have obviously $\mathfrak{a}_\lambda(x)\in\bigcap_{K\in\mathfrak{K}}\mathfrak{a}_\lambda(K)^-$, and consequently $\mathfrak{a}_\lambda^{-1}(U)\in\mathfrak{K}$ for $\mathfrak{a}_\lambda(x)\in U\in\mathfrak{N}_\lambda$, as proved just above. Therefore we have

$$\bigcap_{\nu=1}^{n} \mathfrak{a}_{\lambda_\nu}^{-1}(U_{\lambda_\nu}) \in \mathfrak{K} \quad \text{for } \mathfrak{a}_{\lambda_\nu}(x) \in U_{\lambda_\nu} \in \mathfrak{N}_{\lambda_\nu} \ (\nu=1, 2, \cdots, n).$$

As all $\bigcap_{\nu=1}^{n} \mathfrak{a}_{\lambda_\nu}^{-1}(U_{\lambda_\nu})$ for every finite number of sets $U_{\lambda_\nu} \in \mathfrak{N}_{\lambda_\nu}$ $(\nu=1, 2, \cdots, n)$ constitute a neighbourhood system of the weak topology of R, we conclude that $x \in K^-$ for all $K \in \mathfrak{K}$. For any system of closed sets $\mathfrak{F}$ subject to the condition (I), we can find by the maximal theorem a maximal system $\mathfrak{K}$ subject to the condition (I) such that $\mathfrak{K} \supset \mathfrak{F}$, and for such $\mathfrak{K}$ we have

$$\bigcap_{K \in \mathfrak{F}} K \supset \bigcap_{K \in \mathfrak{K}} K^- \neq \phi,$$

as proved just above. Thus the weak topology of R is compact.

Let S be a topological space with topology $\mathfrak{T}$. For every point $a \in S$, we define a closed set a^* as

$$a^* = \bigcap_{a \in U \in \mathfrak{T}} U^-.$$

With this definition we have obviously: $\{a\}^- \subset a^*$, and $b \in a^*$ implies $a \in b^*$.

Theorem 3. *If the weak topology of R by $\mathfrak{a}_\lambda$ $(\lambda \in \Lambda)$ is compact, then for any system of points $a_\lambda \in S_\lambda$ $(\lambda \in \Lambda)$ subject to the condition* (F) *we can find a point $x \in R$ for which $\mathfrak{a}_\lambda(x) \in a_\lambda^*$ for all $\lambda \in \Lambda$.*

Proof. For a system of points $a_\lambda \in S_\lambda$ $(\lambda \in \Lambda)$ subject to the condition (F), we have

$$\bigcap_{\lambda \in \Lambda} \ \bigcap_{a_\lambda \in U \in \mathfrak{N}_\lambda} \mathfrak{a}_\lambda^{-1}(U)^- \neq \phi,$$

because R is compact by assumption. For a point $x \in R$ such that

$$x \in \mathfrak{a}_\lambda^{-1}(U)^- \quad \text{for all } a_\lambda \in U \in \mathfrak{N}_\lambda, \ \lambda \in \Lambda,$$

as $\mathfrak{a}_\lambda^{-1}(U)^- \subset \mathfrak{a}_\lambda^{-1}(U^-)$ (cf. TLTS §16, Theorem 3), we have $\mathfrak{a}_\lambda(x) \in U^-$ for all $a_\lambda \in U \in \mathfrak{N}_\lambda$, and hence $\mathfrak{a}_\lambda(x) \in a_\lambda^*$ for all $\lambda \in \Lambda$.

Finally we consider the topologies of S for which $\{a\}^- = a^*$ for every point $a \in S$. We can prove easily:

Lemma. *$\{a\}^- \ni b$ implies always $\{b\}^- \ni a$, if and only if $a \in U \in \mathfrak{T}$ implies $\{a\}^- \subset U$.*

If $\{a\}^- = a^*$ for every point $a \in S$, then for any point $b \,\overline{\in}\, \{a\}^-$ we can find $U \in \mathfrak{T}$ such that $a \in U$ and $b \,\overline{\in}\, U^-$, and hence by Lemma $\{a\}^- \subset U$ and $\{b\}^- \subset U^{-\prime}$. Thus we have

Theorem 4. *We have $\{a\}^- = a^*$ for every point $a \in S$, if and only if the partition space of S by the partition $\{a\}^-$ $(a \in S)$ is a Hausdorff space.*

Remark 1. The condition about point system in Theorem 2 is not necessary. Because, let $\{a, b\}$ be a topological space with the topology: $\{a, b\}$, $\{a\}$, ϕ. The point set $\{a\}$ is obviously compact by the relative topology, but $a \,\overline{\in}\, \{b\}^- = \{b\}$.

Remark 2. The condition in Theorem 3 is not sufficient. Because, let $\{0, 1, 2, \cdots\}$ be a topological space with a neighbourhood system: $\{0, 1, 2, \cdots\}$, $\{n\}$ $(n=1, 2, \cdots)$. This space is obviously compact. It is clear that a point set $\{1, 2, \cdots\}$ is not compact by the relative topology but we have $n \in 0^* = \{0, 1, 2, \cdots\}$ for every $n=1, 2, \cdots$.