100. *Finite-to-one Closed Mappings and Dimension. II*

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In this note[1] our concern is devoted to mappings defined on spaces of positive dimension, though in the previous note [3] we were mainly concerned with mappings defined on 0-dimensional spaces. Theorem 1 below gives an answer for the problem concerning dimension-raising mappings between non-separable metric spaces, which was raised by W. Hurewicz [1] and solved for the case of separable metric spaces by J. H. Roberts [4]. All notations and terminologies used here are the same as in the previous note [3]. A space R has dimension $\leq \aleph_0$, $\dim R \leq \aleph_0$, if R is the countable sum of subspaces R_i with $\dim R_i \leq 0$.

Let R and S be topological spaces. Let $\mathfrak{F}=\{F_\alpha; \alpha \in A\}$ and $\mathfrak{H}=\{H_\alpha; \alpha \in A\}$ be respectively locally finite closed coverings of R and S. Let f be a mapping of R onto S. Let r be a positive integer. If the following conditions are satisfied, $(R, \mathfrak{F}, f)$ is called *the cut of order* r of $(S, \mathfrak{H})$.

(1) For every $\alpha \in A$, $f|F_\alpha$ is a homeomorphism of F_α onto H_α.

(2) If order $(y, \mathfrak{H})$, the number of closed sets of $\mathfrak{H}$ which contain $y \in S$, is greater than r, $f^{-1}(y)$ consists of one and only one point.
If $r_1=$order $(y, \mathfrak{H})$ is not greater than r, $f^{-1}(y)$ consists of exactly r_1 points.

R is called the cut-space of order r obtained from $(S, \mathfrak{H})$. $\mathfrak{F}$ is called the derived covering of order r and f the cut-mapping. We can prove that there exists the cut of order r of $(S, \mathfrak{H})$ for any $(S, \mathfrak{H})$ and r and that the cut is essentially unique.

Let R_0 be a metric space with $\dim R_0=n$, $0<n<\infty$. Let m be an arbitrary integer with $0 \leq m < n$. We shall now construct a metric space T with $\dim T=m$ and a closed mapping π_0 of T onto R_0 such that for every point p of R_0 $\pi_0^{-1}(p)$ consists of at most $n-m+1$ points.

By [2] or [3] there exist $\lim A_i = \lim\{A_i, f_{i+1,i}\}$, where A_i are discrete spaces of indices, and a sequence of locally finite closed coverings $\mathfrak{F}_{0i}=\{F(0, \alpha_i); \alpha_i \in A_i\}$, $i=1, 2, \cdots$, which satisfy the following conditions.

(1) The diameter of each set of $\mathfrak{F}_{0i} < 1/i$.

(2) The order of every $\mathfrak{F}_{0i} \leq n+1$.

1) The detail of the content of the present note will be published in another place.

(3) For any i and any $\alpha_i \in A_i$,[2]

$$F(0, \alpha_i)=\cup\{F(0, \alpha_{i+1}); \alpha_{i+1}\in A_{i+1}, f_{i+1,i}(\alpha_{i+1})=\alpha_i\}.$$

Let $(S_{i1}, \mathfrak{H}_{iii}=\{H(i, i, \alpha_i); \alpha_i \in A_i\}, h_{i1})$ be the cut of order $r=n-m+1$ of $(R_0, \mathfrak{F}_{0i})$. Let

$$H(i, i, \alpha_j)=\cup\{H(i, i, \alpha_i); f_{ij}(\alpha_i)=\alpha_j\} \quad \text{if } j<i,$$

and

$$H(i, i, \alpha_j)=h_{i1}^{-1}(F(0, \alpha_j))\cap H(i, i, f_{ji}(\alpha_j)) \quad \text{if } j>i,$$

and then $\mathfrak{H}_{iij}=\{H(i, i, \alpha_j); \alpha_j \in A_j\}$, $j=1, 2, \cdots$, is a sequence of locally finite closed coverings of S_{i1} which satisfies the following conditions.

(1) For any point p of S_{i1} and any neighborhood U of p, there exists $\mathfrak{H}_{iij}$ such that $S(p, \mathfrak{H}_{iij})=\cup\{H(i, i, \alpha_j);\ p\in H(i, i, \alpha_j)\}\subset U$.

(2) For any j and any $\alpha_j \in A_j$,

$$H(i, i, \alpha_j)=\cup\{H(i, i, \alpha_{j+1}); f_{j+1,j}(\alpha_{j+1})=\alpha_j\}.$$

We can construct by a successive process the cuts

$$(S_{ij}, \mathfrak{H}_{i,i-j+1,i-j+1}, h_{ij}), \quad j=2,\cdots, i,$$

and a sequence of locally finite closed coverings

$$\mathfrak{H}_{i,i-j+1,k}=\{H(i, i-j+1, \alpha_k); \alpha_k \in A_k\}, \quad k=1, 2,\cdots,$$

of S_{ij}, $j=2,\cdots, i$, which satisfy the following conditions.

(1) $(S_{ij}, \mathfrak{H}_{i,i-j+1,i-j+1}, h_{ij})$ is the cut of order r of $(S_{i,j-1}, \mathfrak{H}_{i,i-j+2,i-j+1})$.

(2) $H(i, i-j+1, \alpha_k)=\cup\{H(i, i-j+1, \alpha_{i-j+1}); f_{i-j+1,k}(\alpha_{i-j+1})=\alpha_k\}$ if $k>i-j+1$, and

$H(i, i-j+1, \alpha_k)=h_{ij}^{-1}(H(i, i-j+2, \alpha_k))\cap H(i, i-j+1, f_{k,i-j+1}(\alpha_k))$ if $k<i-j+1$.

Then the following conditions are satisfied.

(1) For any point p of S_{ij} and any neighborhood U of p, there exists $\mathfrak{H}_{i,i-j+1,k}$ such that $S(p, \mathfrak{H}_{i,i-j+1,k})\subset U$.

(2) For any k and any $\alpha_k \in A_k$,

$$H(i, i-j+1, \alpha_k)=\cup\{H(i, i-j+1, \alpha_{k+1}); f_{k+1,k}(\alpha_{k+1})=\alpha_k\}.$$

Let $R_i=S_{ii}$, $F(i, \alpha_j)=H(i, 1, \alpha_j)$ and $\mathfrak{F}_{ij}=\{F(i, \alpha_j); \alpha_j \in A_j\}$. Then there is a mapping $g_{i+1,i}$ of R_{i+1} onto R_i such that $g_{i+1,i}|F(i+1, \alpha_{i+1})$ is a homeomorphism of $F(i+1, \alpha_{i+1})$ onto $F(i, \alpha_{i+1})$ for every $\alpha_{i+1}\in A_{i+1}$, $i=0, 1, 2,\cdots$. Let $T=\lim\{R_i, g_{i+1,i}\}$ and π_i be a natural projection of T onto R_i, $i=0, 1, 2,\cdots$.

We can prove that the projection π_0 of T onto R_0 is a closed mapping such that for every point y of R_0 the inverse image $f^{-1}(y)$ consists of at most $r=n-m+1$ points. Moreover we can prove that T thus constructed is a metric space with $\dim T=m$.

Theorem 1. *Let R_0 be a metric space. Then* $\dim R_0\leq n(<\infty)$ *if and only if R_0 is the image of a metric space T with* $\dim T\leq m(\leq n)$ *under a closed mapping π_0 such that $\pi_0^{-1}(y)$ consists of at most $n-m+1$ points for every $y\in R_0$.*

Theorem 2. *Let R be an inverse limiting space of a sequence of*

2) In the following α_j always indicates the element of A_j.

metric spaces R_i, $i=1, 2, \cdots$. *Then we have*

$$\dim R \leq \lim\ \inf \dim R_i.$$

Theorem 3. *Let* R *be an inverse limiting space of a system of compact Hausdorff spaces* R_α, $\alpha \in A$. *Then we have*

$$\dim R \leq \sup_\alpha \dim R_\alpha.$$

Theorem 4.[3] *Let* R *and* S *be metric spaces and* f *a closed mapping of* R *onto* S. *If the boundary of* $f^{-1}(y)$ *consists of exactly* $k(<\infty)$ *points for every point* $y \in S$, *then we have* $\dim S \leq \dim R$. *If moreover* $\dim S < \dim R < \infty$, *there exists* $y \in S$ *such that* $\dim f^{-1}(y) = \dim R$.

Theorem 5. *Let* R *and* S *be metric spaces and* f *a closed mapping of* R *onto* S. *If the boundary of* $f^{-1}(y)$ *is finite for every point* $y \in S$ *and* $\dim R \leq \aleph_0$, *then we have* $\dim S \leq \aleph_0$.

Theorem 6. *Let* R *and* S *be paracompact Hausdorff spaces and* f *an open mapping of* R *onto* S. *If* $f^{-1}(y)$ *is finite for every point* $y \in S$, *then we have* $\dim R = \dim S$. *If moreover* R *and* S *are hereditarily paracompact, we have* $\operatorname{Ind} R = \operatorname{Ind} S$, *where* $\operatorname{Ind} R$ *is the large inductive dimension of* R *defined by means of the separation of closed sets.*

References

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[5] J. Suzuki: Note on a theorem for dimension, Proc. Japan Acad., **35**, 201–202 (1959).

3) The main part of this theorem was proved independently by J. Suzuki [5].