99. A New Characterization of Paracompactness

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The present note deals with another characterization of paracompactness of regular spaces and linearly ordered spaces. Our result concerning regular spaces is closely related to that of Kelley [2, p. 156], which asserts that if X is a regular space, then X is paracompact if and only if each open covering of X is even. Moreover our result concerning linearly ordered spaces is a generalization of that of Gillman and Henriksen [1] which asserts that a linearly ordered Q-space is paracompact.

1. For an open covering $\mathfrak{U}=\{G_\gamma \mid \gamma \ni \Gamma\}$ of a T-space X and a neighborhood U of the diagonal Δ of the product space $X\times X$, let A_U be the closure of the set of all points x such that $U(x)$ is not contained in every member G_γ of $\mathfrak{U}$, where $U(x)=\{y \mid (x, y)\in U\}$. It is clear that if $U\subset V$, then $A_U\subset A_V$. Set $H_U=A_U^c$. Then H_U is an open set of X, and if $U\subset V$, then $H_U\supset H_V$. Now let $\mathfrak{F}$ be the family of all the neighborhoods of the diagonal Δ of $X\times X$. Then we have the following

Lemma. *If X is a regular space, then $\{H_U \mid U\in\mathfrak{F}\}$ is an open covering of X.*

Proof. If there is an A_U such that $A_U=\phi$, then for such A_U we have $H_U=A_U^c=X$. Now suppose that $A_U\neq\phi$ for every $U\in\mathfrak{F}$. For any point x of X there is a G_γ such that $x\in G_\gamma$, and, since X is regular, there are open sets H and K such that

$$G_\gamma\supset\overline{H}\supset H\supset\overline{K}\supset K\ni x.$$

Let us put $U=(H\times H)\cup(K^c\times K^c)$. Then U is a neighborhood of the diagonal Δ, and $U(x)=H\subset G_\gamma$. Moreover for any point y of K, $U(y)=H\subset G_\gamma$. Therefore x is contained in $H_U=A_U^c$. Thus $\{H_U \mid U\in\mathfrak{F}\}$ is an open covering of X. This completes the proof of the lemma.

In case X is regular, we call $\tilde{\mathfrak{U}}=\{H_U \mid U\in\mathfrak{F}\}$ an open covering of X derived from an open covering $\mathfrak{U}$ of X.

Theorem 1. *If X is a regular space, then the following statements are equivalent:*

(1) *X is paracompact.*

(2) *Every open covering $\tilde{\mathfrak{U}}$ of X derived from any open covering $\mathfrak{U}$ of X has a finite subcovering.*

Proof. (1)$\rightarrow$(2). Since X is paracompact, any open covering $\mathfrak{U}$ is even, that is, there is a $U_0\in\mathfrak{F}$ such that for each x $U_0(x)$ is contained in some member of $\mathfrak{U}$. This shows that $A_{U_0}=\phi$, i.e. $H_{U_0}=X$. Hence

$\mathfrak{U}$ has a finite subcovering. (2)→(1). Let $\mathfrak{U}$ be any open covering of X. Then an open covering $\tilde{\mathfrak{U}}=\{H_U \mid U\in\mathfrak{F}\}$ derived from $\mathfrak{U}$ has a finite subcovering $\{H_{U_i} \mid i=1, 2,\cdots, n\}$. Let $V=U_1\cap U_2\cap\cdots\cap U_n$. Since $H_{U_i}\subset H_V$ $(i=1, 2,\cdots, n)$, we obtain $H_V=X$, i.e. $A_U=\phi$. This shows that for each x $V(x)$ is contained in some member of $\mathfrak{U}$. Hence $\mathfrak{U}$ is even. Therefore X is paracompact.

2. In the sequel we state a generalization of a result of Gillman and Henriksen [1] concerning linearly ordered spaces. It is well known that a linearly ordered space with the interval topology is normal. Hereafter we use the same terminologies as in [1].

Theorem 2. *If X is a linearly ordered space with the interval topology, then the following statements are equivalent*:

(1) *X is paracompact.*

(2) *X has a complete structure.*

Proof. Since the strongest uniformity of a paracompact space is complete, we prove only that (2) implies (1). Since a linearly ordered space is paracompact if and only if every gap of X is a Q-gap by [1, Theorem 9.5], we show that every gap of X is a Q-gap. Now suppose that there is a gap u which is not a Q-gap from the left. If gX is a complete uniform structure of X and $\{\mathfrak{U}_\alpha \mid \alpha\in A\}$ is a uniformity of gX, then for each α the open covering $\mathfrak{U}_\alpha$ has a locally finite open refinement $\mathfrak{B}_\alpha=\{H_\gamma^\alpha \mid \gamma\in\Gamma^\alpha\}$, since each $\mathfrak{U}_\alpha$ is a normal covering of a Hausdorff space X. Let y_0 be the first point or gap of X and let $J=[y_0, u)$. Then the open covering $\mathfrak{B}'_\alpha=\{H_\gamma^\alpha\cap J \mid \gamma\in\Gamma^\alpha\}$ of J is locally finite. Therefore by [1, Lemma 9.4] which asserts that any open covering of J which does not cover the gap u (that is not a Q-gap from the left) is not locally finite, the gap u is to be covered by $\mathfrak{B}'_\alpha$. Hence for each α, we can select an open set $H_{\gamma(\alpha)}^\alpha\in\mathfrak{B}_\alpha$ such that $H_{\gamma(\alpha)}^\alpha\cap J$ covers the gap u. If we set $\mathfrak{F}=\{H_{\gamma(\alpha)}^\alpha\cap J \mid \alpha\in A\}$, then it is clear that $\mathfrak{F}$ is a Cauchy filter in gX. Hence it follows from (2) that $\mathfrak{F}$ converges to a point x_0 of X. The point x_0 is contained in J, because J is open and closed in X. Let z be a point of the open interval (x_0, u). Then the open interval $[y_0, u)$ contains a member of $\mathfrak{F}$, since $\mathfrak{F}$ converges to x_0. This contradicts the fact that any member of $\mathfrak{F}$ covers the gap u. In case there is a gap v which is not a Q-gap from the right, we obtain the contradiction similarly. This completes the proof of the theorem.

References

[1] L. Gillman and M. Henriksen: Concerning rings of continuous functions, Trans. Amer. Math. Soc., **77**, 340-362 (1954).

[2] J. L. Kelley: General Topology, New York (1955).