98. On Locally Q-complete Spaces. III

By Takesi ISIWATA
Tokyo Gakugei University, Tokyo

(Comm. by K. KUNUGI, M.J.A., Oct. 12, 1959)

We assume always that $X^{*)}$ is locally Q-complete but not a Q-space. Then there are one-point Q-completions of X [2]. In this paper, we shall investigate some properties of one-point Q-completions of X. We noticed, in [2], that X is open in νX and $X\smile(\nu X-X)^{\beta}$ is a Q-space. We have similarly that if B is any compact subset in $\beta X-X$ which contains $\nu X-X$ then the space $X\smile B$ is also a Q-space, and moreover the space Z obtained from $X\smile B$ by contracting B to a point in B is a one-point Q-completion (Theorem 1 in [2]). In the following, we shall prove that any one-point Q-completion of X is given as an image of a space $X\smile B$ under a continuous mapping φ such that $\varphi\,|\,X$ is a homeomorphism which leaves every point of X invariant where B is some compact subset in $\beta X-X$ which contains $\nu X-X$.

Lemma 1. *Suppose that $Z=X\smile\{p\}$ is a one-point Q-completion of X. Then there is a continuous mapping ψ of νX onto Z such that $\psi(\nu X-X)=\{p\}$, $\psi(x)=x$ for every $x\in X$ and $\psi\,|\,X$ is a homeomorphism.*

Proof. X is considered as a uniform space X_1 with the structure generated by $C=\{f\,|\,X;\ f\in C(Z)\}$ and Z becomes a completion of X_1. On the other hand, X may be considered as a uniform space X_2 with the structure generated by $C(X)$. Since $C(X)\supset C$ and the identical mapping i is uniformly continuous, i has a continuous extension ψ of νX to Z. Hence, to prove Lemma, it is sufficient to show that $\psi(\nu X-X)=p$. Suppose that $\{a_\alpha;\ a_\alpha\in X\}\to a\in\nu X-X$ and $\psi(a)=b\in X\subset Z$. We take an open neighborhood V(in Z) of b which does not contain p. $i^{-1}(V)$ is open in νX because X is open in νX. By the assumption, for some index α_0, $\alpha>\alpha_0$ implies $\psi(a_\alpha)=i(a_\alpha)\in V$, and hence $i^{-1}(V)\ni a_\alpha$ for $\alpha>\alpha_0$. This is a contradiction. We have therefore that $\psi(\nu X-X)=p$.

For any point $x\in Z$, let us put $B(x)=\frown\overline{\psi^{-1}(V)}$ (in βX) where V runs over all neighborhoods (in Z) of x. Since $\psi(\nu X-X)=p$. $B(p)$ is a compact subset containing $\nu X-X$.

Lemma 2. *$B(x)=\{x\}$ for any $x\in X\subset Z$ and $B(p)\subset\beta X-X$.*

Proof. For any point $y\in X\subset Z$, there is an open neighborhood U (in Z) of $y\in X\subset Z$ which is disjoint from some neighborhood (in Z) of p. We have therefore $B(p)\not\ni y$, which implies that $B(p)\subset\beta X-X$. Simi-

*) A space X considered here is always a completely regular T_1-space, and other terminologies used here, for instance "Q-completion," are the same as in [2,3].

larly we have $B(x)=\{x\}$ for any point $x\in X\subset Z$.

We define a mapping φ of $X\smile B(p)$ onto Z by

$$\varphi(x)=\begin{cases}\psi(x) & \text{for } x\in\nu X,\\ p & \text{for } x\in B(p).\end{cases}$$

Then φ is a continuous mapping and $X\smile B(p)$ is the largest subspace of βX on which ψ has a continuous extension (Theorem 2.1 in [1]).

Now suppose that Z is a one-point Q-completion of X which is obtained form $X\smile B$ by contracting B to a one point p where B is a compact subset, containing $\nu X-X$, contained in $\beta X-X$. Then we have

Lemma 3. $B(p)=B$.

Proof. Let ψ be a mapping from $X\smile B$ onto Z and φ a mapping mentioned above. $\psi\,|\,\nu X$ is a continuous mapping from νX on Z, and $X\smile B(p)$ is the largest subspace of $\beta(X)$ on which $\psi\,|\,\nu X$ has a continuous extension. Therefore we have $B(p)\supset B$. If there is a point $b\in B(p)-B$, then there is a directed set $\{a_\alpha;\ \alpha\in P\}$ in X which converges to b in $X\smile B(p)$, but does not converge in $X\smile B$. In $X\smile B(p)$, there are disjoint open subsets U and V such that $U\supset B$, $V\ni p$ and $\overline{U}\frown\overline{V}=\theta$. Then $(X\smile B)\frown\overline{U}$ is disjoint from $(X\smile B)\frown\overline{V}$ and their images under ψ are disjoint from each other. This shows that $\psi(a)_\alpha$ deos not converge to p. On the other hand $\psi(a_\alpha)=\varphi(a_\alpha)\rightarrow b$. This is a contradiction, and hence $B=B(p)$.

Suppose that B_1 and B_2 are compact subsets contained in $\beta X-X$ and Z_i is a one-point Q-completion of X obtained from $X\smile B_i$ contracting B_i to a point p_i $(i=1,2)$. As is easily seen from the proof of Lemma 3, under a mapping φ which maps X homeomorphically onto X and which keeps X pointwisely fixed, Z_1 is homeomorphic with Z_2 if and only if $B_1=B_2$.

Let $Q(X)$ be a family of all one-point Q-completions of X. We shall define that for any $Z_1, Z_2\in Q(X)$, Z_1 is equal to Z_2 if and only if there is a homeomorphism from Z_1 on Z_2 which maps X onto X pointwisely fixed.

Theorem 1. *Let X be locally Q-complete but not a Q-space. Then there is a one-to-one correspondence between $Q(X)$ and a set of all compact subsets contained in $\beta X-X$ which contain $\nu X-X$.*

If $Z_\alpha\in Q(X)$ is a continuous image of $X\smile B$ where B is a compact subset contained in $\beta X-X$ containing $\nu X-X$, then we set $B=B_\alpha$. We shall define $Z_\alpha > Z_\beta$ for any $Z_\alpha, Z_\beta\in Q(X)$ if and only if there is a continuous mapping $f_{\alpha\beta}$ from Z_α onto Z_β such that $f_{\alpha\beta}\,|\,X$ is the identical homeomorphism and $f_{\alpha\beta}\ (Z_\alpha-X)=Z_\beta-X$.

Suppose that $B_\alpha\subset B_\beta\subset\beta X-X$ and j is an injection from B_α into B_β and φ_α(or φ_β) is a continuous mapping from $X\smile B_\alpha$(or $X\smile B_\beta$) onto Z_α (or Z_β) respectively such that $\varphi_\alpha(B_\alpha)$ (or $\varphi_\beta(B_\beta))=Z_\alpha-X$(or $Z_\beta-X$), and $\varphi_\alpha\,|\,X$ is a homeomorphism which keeps X pointwisely fixed. It is easily verified that $\varphi_\beta j \varphi_\alpha^{-1} = f_{\alpha\beta}$ is a continuous mapping from Z_α onto Z_β such that $f_{\alpha\beta} | X$ is the identical homeomorphism. Conversely suppose that $Z_\alpha > Z_\beta$, then by the definition of B_α, B_β, and the fact that any open set in Z_β is also open in Z_α, we have $B_\alpha \subset B_\beta$. Therefore the relation " $<$ " in $Q(X)$ is transformed into the inclusion relation among the family of compact subsets contained in $\beta X - X$ containing $\nu X - X$. Therefore $Q(X)$ becomes a lattice. Let Z_{α_1} be a one-point Q-completion of X where $B_{\alpha_1} = (\nu X - X)^\beta$. It is easy to see that $Z_{\alpha_1} > Z_\alpha$ for any $Z_\alpha \in Q(X)$, that is, Z_{α_1} is the largest element 1 in the lattice $Q(X)$ (Z_{α_1} is called a *natural one-point Q-completion* of X (see [2])).

Theorem 2. *If X is locally Q-complete but not a Q-space, then $Q(X)$ is a lattice having the largest element* 1, *in other words, the natural one-point Q-completion is the largest element in $Q(X)$.*

Suppose that X is not locally compact, then there is not an element Z_{α_0} in $Q(x)$ such that $Z_\alpha > Z_{\alpha_0}$ for any $Z_\alpha \in Q(X)$. For, since $\beta X - X$ is not compact, for any $Z_\alpha \in Q(X)$, we have $B_\alpha \neq \beta X - X$, and hence there is a point b in $\beta(X - X) - B_\alpha$. This shows that $Z_\alpha > Z_\beta$ where $B_\beta = B_\alpha \cup \{b\}$. If X is locally compact, it is easy to see that $Z_\alpha > Z_{\alpha_0}$ for any $Z_\alpha \in Q(X)$ where $B_{\alpha_0} = \beta X - X$. Thus Z_{α_0} is the smallest element 0 in the lattice $Q(X)$. Thus we have

Theorem 3. *Let X be locally Q-complete but not a Q-space; then X is locally compact if and only if $Q(X)$ is a lattice having the smallest element* 0.

As an immediate consequence of Theorem 2, if $Q(X)$ is a finite lattice, X must be locally compact. Moreover we can prove, in this case, that $(\nu X - X)^\beta = \beta X - X$. For, suppose the contrary. Since $Y = X \cup (\nu X - X)^\beta$ has the property such that $\beta Y = \beta X = Y \cup D$ where D is a finite set, Y must be pseudo-compact. On the other hand, Y is a Q-space, and hence Y must be compact. This implies that $\beta X = Y$.

Conversely, if X is locally compact and $(\nu X - X)^\beta = \beta X - X$, it is obvious that $Q(X)$ consists of only one element. Thus we see that $Q(X)$ *is a finite lattice if and only if X is locally compact and $\nu X - X$ is dense in $\beta X - X$.*

Finally, we shall consider some subring of $C(X)$ where Z_α is any one-point Q-completion of X. Let $Z_\alpha = X \cup \{p\}$ and $Y = X \cup B$ where $B = B_\alpha$. Now we denote by $C(Z, p)$ the ring consisting of all continuous functions defined on Z which vanish on some neighborhood of p. Any element in $C_B(X)$ is considered as a function in $C(Z, p)$. Conversely it is easy to see that any function in $C(Z, p)$ can be regarded as a function in $C_B(X)$. In [3], we proved that any non-trivial ring homomorphism on $C_B(X)$ is a point ring homomorphism. From these facts, we have that any ring homomorphism φ on $C(Z, p)$ is a point ring homomorphism, that is, i) if φ is not trivial, $\varphi = \varphi_x$, $x \neq p$, ii) if φ is trivial,

$\varphi=\varphi_p$ (we notice that $C(Z, p)$ is a linear subring of $C(Z)$). Thus we have

Theorem 4. *Let X be locally Q-complete but not a Q-space. If Z is any one-point Q-completion of X, then any ring homomorphism on $C(Z, p)$ is a point ring homomorphism φ_x where $Z=X \cup \{p\}$, $x \in X$.*

References

[1] R. H. McDowell: Extension of functions from dense subspaces, Duke Math. Jour., **25**, 297-304 (1958).
[2] T. Isiwata: On locally Q-complete spaces. I, Proc. Japan Acad., **35**, 232-236 (1959).
[3] T. Isiwata: On locally Q-complete spaces. II, Proc. Japan Acad., **35**, 263-267 (1959).