123. On Monotone Solutions of Differential Equations

By Heinrich W. GUGGENHEIMER

(Comm. by K. KUNUGI, M.J.A., Nov. 12, 1959)

In a recent note [1], Professor Iséki proved the following theorem, which we will formulate for one single differential equation: *If the functions $P(t)$, $Q(t)$ are defined and absolutely integrable on an interval $[a, +\infty)$, then any monotone increasing solution $x(t)$ of*

$$\frac{dx}{dt}=P(t)x+Q(t)$$

is bounded on that interval.

It is natural to ask whether a similar theorem may hold for an equation

$$\frac{dx}{dt}=\sum_{m=0}^{n}p_m(t)x^m \tag{1}$$

with suitable conditions on the coefficients $p_m(t)$. It is immediately seen that if the leading coefficient $p_n(t)$ is integrable, no similar result can hold, since

$$\frac{dx}{dt}=\frac{1}{t^2}x^2$$

has the unbounded solution $x=t$. So one may try to get the desired result from the opposite condition: $P_n(t)$ not integrable on $[a, +\infty)$, since the example

$$\frac{dx}{dt}=\frac{1}{t}x^2$$

has the monotone solution $x=-(\log t)^{-1}$, bounded by 0.

It turns out that this special situation is the general one, since we have

Theorem 1. *If the functions $p_m(t)$, $m=0,\cdots,n$ are defined on an interval $[a, +\infty)$, and if*

i) $\int_a^\infty p_n(u)du=+\infty,\quad p_n(t)\geq 0,\qquad (t\geq T_0)$

ii) $p_m(t)\geq 0,\quad 0\leq m\leq n-1,\qquad (t\geq T_0)$

then any monotone increasing solution $x(t)$ of (1)

a) *either is bounded by zero*

b) *or* $\lim_{t\to\infty}\frac{x(t)}{\int_T^t p_n(u)du}=\infty.$

Proof. By hypothesis i), $p_n(t)$ is positive for $t>T_0$. Now suppose that there is a T (it may be taken $>T_0$) with $x(T)>0$, $x(t)$ being a monotone increasing solution of (1). From (1) we have by integration

$$x(t)-x(T)=\int_T^t x^n(u)p_n(u)du+\sum_{m=0}^{n-1}\int_T^t p_m(u)x^m(u)du \tag{2}$$

hence

$$x^n(t^*)\int_T^t p_n(u)du<x(t) \qquad \text{for a } t^*,\ T\leq t^*\leq t,$$

or

$$x^n(t^*)<\frac{x(t)}{\int_T^t p_n(u)du}.$$

From this inequality, both parts of the theorem follow at once.

Theorem 2. *If the functions $p_m(t)$, $m=0,\cdots,n$ are defined on an interval $[a,+\infty)$, and if*

i) $$\left|\int_a^\infty p_n(u)du\right|=+\infty$$

ii) $$\int_a^\infty |p_m(u)|du<\infty \qquad 0\leq m\leq n-1$$

then any monotone increasing solution of (1) *is bounded by zero.*

As in the proof of Theorem 1, let us suppose $x(T)>0$. Then we have from (2), using the first mean value theorem,

$$x^n(t^*)\left|\int_T^t p_n(u)du\right|\leq x(t)+x(T)+\sum_{m=0}^{n-1}x^m(t_m)\left|\int_T^t p_m(u)du\right|,\quad T\leq t_m<t,$$

and by the monotonicity, for T big enough to fulfill

$$\int_T^\infty |p_m(u)|du<\varepsilon$$

we have

$$x^n(t^*)\left|\int_T^t p_n(u)du\right|<x(t)+x(T)+(n-1)\varepsilon x^{n-1}(t)+\varepsilon. \tag{3}$$

If $x(t)$ is assumed bounded, a contradiction follows at once by hypothesis i). If $x(t)$ is not bounded, then we may assume $x(T)\geq 1$, and (3) gives, dividing by $x^{n-1}(t)\left|\int_T^t p_n(u)du\right|$,

$$x(t)<\frac{1+n\varepsilon+x(T)}{\left|\int_T^t p_n(u)du\right|}$$

from which a contradiction follows, also by hypothesis i). This completes the proof.

Remark. Defining a vector of functions

$$x(t)=\begin{pmatrix} x_1(t)\\ \vdots \\ x_s(t)\end{pmatrix},\qquad x^m(t)=\begin{pmatrix} x_1^m(t)\\ \vdots \\ x_s^m(t)\end{pmatrix}$$

and replacing the functions $p_n(t)$ by matrices, the theorem is generalized at once to systems of differential equations.

Reference

[1] K. Iséki: A remark on monotone solutions of differential equations, Proc. Japan Acad., **35**, 370–371 (1959).