20. The Thickening of Combinatorial n-manifolds in (n+1)-space

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The sets which come into consideration are all to be polyhedral in some Euclidean space and manifolds, cells, spheres are to be *combinatorial*; all homeomorphisms, imbeddings are to be *piecewise linear*.

The regular neighborhood is originally defined by J. H. C. Whitehead,[1] which is not necessary the neighborhood in the set theoretic sense. We put some restrictions to it as follows.

Definition. Let P be a finite polyhedron imbedded in an m-manifold W *without boundary*. The *regular neighborhood* $U(P, W)$ of P in W means an m-manifold contained in W and *containing* P *in the interior*, which contracts geometrically into P.

Then the results of Whitehead imply the following

Theorem 1. *Let P be a finite polyhedron imbedded in a manifold W without boundary. Then for any two regular neighborhoods $U_1(P, W)$ and $U_2(P, W)$ of P in W there is a homeomorphism onto $\psi: W \to W$ such that $\psi(U_1(P, W)) = U_2(P, W)$ and $\psi \mid P =$ identity where ψ is an orientation preserving homeomorphism if W is orientable.*

The *combinatorial version of the Schönflies conjecture for dimension n* is the following statement: *Let an $(n-1)$-sphere S^{n-1} be imbedded in Euclidean n-space R^n. Then the closure of the bounded component of $R^n - S^{n-1}$ is an n-cell.*

This has been affirmatively proved[2] for $n \leqq 3$. Theorem 1 enables us to prove the following

Theorem 2. *Let a compact, n-manifold M_i without boundary be imbedded into an orientable, oriented $(n+1)$-manifold W_i without boundary, $i=1, 2$. Let $U(M_i, W_i)$ be a regular neighborhood of M_i in W_i and $\phi: M_1 \to M_2$ be a homeomorphism onto.*

Suppose that the combinatorial version of the Schönflies conjecture is true for dimension $\leqq n$.

Then there is a homeomorphism onto $\psi: U(M_1, W_1) \to U(M_2, W_2)$ such that $\psi \mid M_1 = \phi$ and such that the oriented image of oriented

1) J. H. C. Whitehead: Simplicial spaces, nuclei and m-groups, Proc. London Math. Soc., **45**, 243-327 (1935).

2) J. W. Alexander: On the subdivision of 3-space by a polyhedron, Proc. Nat. Sci. U. S. A., **10**, 6-8 (1924); W. Graeub: Die Semilineare Abbildungen, Sitz-Ber. d. Akad. Wissensch. Heidelberg, 205-272 (1950); E. E. Moise: Affine structures in 3-manifolds. II. Positional properties of 2-spheres, Ann. of Math., **55**, 172-176 (1952).

$U(M_1, W_1)$ is the oriented $U(M_2, W_2)$ where the orientation of $U(M_i, W_i)$ is induced by that of W_i.

In the proof of Theorem 2 we make extensive use of combinatorial methods and results of V. K. A. M. Gugenheim.[3)]

As consequents of Theorem 2 we have the following theorems.

Theorem 3. *Let a compact, orientable n-manifold M without boundary be imbedded in an orientable $(n+1)$-manifold W without boundary. Let $U(M, W)$ be a regular neighborhood of M in W.*

Suppose that the combinatorial version of the Schönflies conjecture is true for dimension $\leqq n$.

Then there is a homeomorphism into $\theta: M\times J\to W$ such that $\theta(x, 0)=x$ for all $x\in M$ and such that $\theta(M\times J)=U(M, W)$, where J is the interval $-1\leqq s\leqq 1$.

Theorem 4. *Let M be a compact, orientable n-manifold without boundary imbedded in an orientable $(n+1)$-manifold W without boundary. Let $\phi: M\to M$ be a homeomorphism which is onto isotopic to the identity.*

Suppose that the combinatorial version of the Schönflies conjecture is true for dimension $\leqq n$.

Then there is an orientation preserving homeomorphism onto $\psi: W\to W$ such that $\psi\,|\,M=\phi$.

3) V. K. A. M. Gugenheim: Piecewise linear isotopy and embedding of elements and spheres (I), Proc. London Math. Soc., **3**, 29-53 (1953); (II), ibid., **3**, 129-152 (1953).