52. Characterizations of Spaces with Dual Spaces

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In the following we assume that spaces considered here are always completely regular and continuous functions are real-valued one. Let $X^*=\beta X-X$. We shall say that X has a dual space X^* if there is a homeomorphism of $\beta(X^*)$ onto βX which keeps X^* pointwisely fixed.[1] Then we write $X^{**}=(X^*)^*(=\beta(X^*)-X^*)=X$ or $\beta X=\beta(X^*)$. This notations may be justified by the properties A), B) and C) in §1. A subset B of X is said to be *inessential to* X if any *bounded* continuous function defined on $X-B$ is continuously extended over X. In §2 we shall show that if X has a dual space, then every compact subset of X is inessential to X and every finite subset of βX is inessential to βX. Using this results, we shall prove that X has a dual space if and only if every proper open subset of X whose complement is compact has a dual space.[2] We have given in [3] a stonean space with a dual space. In §3, we shall give examples of spaces with dual spaces among spaces of the following types: i) pseudo-compact spaces, ii) countably compact, Σ-product spaces, iii) countably compact, non-paracompact, normal spaces which have a uniform structure by the family of neighborhoods of the diagonal of product with itself, and iv) countably compact, non-normal spaces.

1. The proofs of the following properties are obvious.

A) Let Z and X be given spaces and let Y be a dense subset of Z. If two homeomorphisms φ and ψ from Z onto X coincide with each other on Y, then $\varphi(z)=\psi(z)$ for every $z\in Z$.

Let φ be a homeomorphism from $\beta(X^*)$ onto βX which keeps X^* pointwisely fixed.

B) If X has a dual space X^*, then every bounded continuous function f^* on X^* has a continuous extension $f=F\circ\varphi^{-1}$ over βX where F is a continuous extension of f^* over $\beta(X^*)$.

C) In B), let g be a bounded continuous function on X and g^* be

1) The definition, in [3], of a dual space (the first row of p. 148 and the last row of p. 160) seems to be ambiguous, but the progression of arguments, in [3], with respect to a dual space was set in the sense of this paper.

2) This characterization may be of interest in view of the fact that the following conditions are equivalent for any X: i) X is a stonean space with a dual space, ii) any proper open subspace U of X has a dual space and $X-U$ is inessential to X, and iii) any proper dense subspace of X has a dual space. This fact is essentially proved in [3, Th. 12] (but with an inexact statement).

its continuous extension over βX. If G is a continuous extension of $g^*|X^*$ over $\beta(X^*)$, then we have $G\circ\varphi^{-1}=g$.

2. Lemma 1. *If X has a dual space, then we have* i) *every point of X has no compact neighborhoods,* ii) *every compact subset of X is inessential to X, and* iii) *every finite subset of βX is inessential to βX.*

Proof. i) Obvious.

ii) Let B be a compact subset of X, $\{U_\alpha;\alpha\in\Gamma\}$ be a base of neighborhoods, in βX, of B and let f be a bounded continuous function on $X-B$. Since βX is normal, there is a continuous function g_α for each $\alpha\in\Gamma$ such that $g_\alpha=0$ on a neighborhood V_α, in βX, contained in U_α, $g_\alpha=1$ on $\beta X-U_\alpha$ and $0\leq g\leq 1$. We put $f_\alpha=fg_\alpha$ on $X-B$ and $f_\alpha=0$ on $X\frown V_\alpha$ for each α. It is obvious that every f_α is continuous on X. Let f_α^* be a continuous extension of f_α over βX. Now let us put $F(x)=\sup_{\alpha\in\Gamma} f_\alpha^*(x)$ for each $x\in\beta X$. For any point $x\in\beta X-B$, since $\{U_\alpha\}$ is a base of neighborhoods, in βX, of B, we have $f_\alpha^*(x)=f_\beta^*(x)$ on some neighborhood (in βX) of x for $\alpha,\beta>\alpha_0$ where α_0 is a suitable index in Γ. This means that F is continuous on $\beta X-B$. By the method of construction of f_α, it is obvious that $f=F|(X-B)$. Thus $F_1=F|(\beta X-B)$ is a continuous extension of f over $\beta X-B$. Since $\beta X-B\supset X^*$ and X has a dual space X^*, F_1 has a continuous extension over βX, and hence over X. Therefore B is inessential to X.

iii) The proof is obtained by the analogous method as used in the proof of ii) (or see [6]).

As easily seen from the proof of ii), any bounded continuous function on $X-F$ has a continuous extension over $\beta X-\bar{F}$ (in βX) for any closed subset F of X even if X has not a dual space.

We shall introduce an order relation in a family of subsets of βX by the inclusion relation. Then for any point $z\in\beta X-X$, by Lemma 1 it is easily seen that $X\smile\{z\}$ has a dual space and $\beta(X\smile\{z\})=\beta X$. Thus we have

Theorem 1. *If a compact space Z is a Čech compactification of a space X with a dual space, then there are no maximal subspaces of Z with dual spaces whose Čech compactifications are Z.*

Theorem 2. *The following conditions are equivalent for any space X:*

i) *X has a dual space,*

ii) *every proper open subspace of X whose complement is compact has a dual space,*

iii) *every point of X has no compact neighborhoods and any proper open subset of X whose complement is compact is inessential to βX,*

iv) *every point of X has no compact neighborhoods and X is* *inessential to every compact space Z containing X as a dense subset.*

Proof. (i)$\leftrightarrow$(iii)$\leftrightarrow$(iv) was obtained by Theorems 10 and 11 in [3].

(i)$\rightarrow$(ii). Suppose that U is open in X and $B=X-U$ is compact. Since B is compact and every point of X has no compact neighborhoods, it is easy to see that $\overline{U}(\text{in } \beta X)=\overline{(\beta X-U)}(\text{in } \beta X)=\beta X$. By Lemma 1, any bounded continuous function on U has a continuous extension over X, and hence βX. Conversely, let f be a bounded continuous function on $\beta X-U$. Then, since $\beta X-U\supset\beta X-X$ and X has a dual space, $f|(\beta X-X)$ has a continuous extension over βX, and hence over U. Thus U has a dual space.

(ii)$\rightarrow$(i). It is obvious that every point of X has no compact neighborhoods. We shall first show that a dual space of U is $W=\beta X-U$ for every open subspace U of X whose complement B is compact. Let f be a bounded continuous function on U. Then by the assumption $V=X-\{p\}$, $p\in X-B$, has a dual space. The $f|(V-B)$ has a continuous extension over V by Lemma 1. This means that f is continuously extended over X, and hence βX. Therefore a dual space of U is W.

Let g be a bounded continuous function on X^*. Then X^* is an open subspace of W whose complement is a compact set B. Thus g has a continuous extension over W by Lemma 1. Since W is a dual space of U, g can be continuously extended over U. This means that f has a continuous extension over X, that is, X has a dual space.

Corollary.[3] *Let X be a space with a dual space: then we have*

i) *if $M\times X$ has a dual space for a compact space M, then X^* is pseudo-compact,*

ii) *if X is pseudo-compact, then $M\times X$ has a dual space for any compact space M if and only if X^* is pseudo-compact,*

iii) *if X has a dual space homeomorphic with itself,*[4] *then $M\times X$ has a dual space for any compact space M if and only if X is pseudo-compact.*

Proof. Since X has a dual space, it is easily seen that every point of $M\times X$ has no compact neighborhoods. Suppose that $M\times X$ has a dual space. $M\times\beta X$ is a compactification of $M\times X$. By (iv) of Theorem 2, any bounded continuous function on $M\times X^*$ ($=M\times\beta X-M\times X$) can be continuously extended over $M\times\beta X$. Thus we have $\beta(M\times X^*)=M\times\beta X$. By Glicksberg's theorem (see (G1) in § 3 below) $M\times X^*$ is pseudo-compact, and hence X^* is also pseudo-compact.

(ii) and (iii). These are obvious from (i) and Glicksberg's theorem (G1).

3) We assume, in this corollary, that compact spaces have infinitely many points.

4) An example of such a space X is obtained by setting X a disjoint union of open sets Y and Y^* where Y is a space with a dual space.

3. In this section we shall give examples of spaces with dual spaces. Let $\{X_\alpha\}$ be an uncountable set of spaces and $Z=\boldsymbol{P}_\alpha X_\alpha \ni b=(b_\alpha)$. A subset of Z, denoted by Σ_b, is called a Σ-product with a base point b if Σ_b consists of all points $z=(z_\alpha)$ with $z_\alpha \neq b_\alpha$ for at most countably many α. Moreover we denote by Σ^b a set consisting of all points $z=(z_\alpha)$ of Z with $z_\alpha=b_\alpha$ for at most countably many α.

Glicksberg and Corson have proved the following theorems.

(G1) [1, Theorem 1]. Suppose that every X_α is compact and $\boldsymbol{P}_{\alpha\neq\alpha_0} X_\alpha$ is infinite for every α_0. Then $\beta Z=\boldsymbol{P}_\alpha(\beta X_\alpha)$ if and only if Z is pseudo-compact.

(G2) [1, Theorem 2]. If each X_α is compact and has at least two points, then $\beta(\Sigma_b)=Z$ for every $b\in Z$.

(C1) [2, Theorem 1]. If each X_α is a complete metric space, then a Σ-product Σ_b is normal for every $b\in Z$.

(C2) [2, Theorem 3]. If each X_α is a complete separable metric space, then any Σ-product Σ_b has a uniform structure which is the family of neighborhoods of the diagonal of $\Sigma_b\times\Sigma_b$ for every $b\in Z$.

From these theorems we shall construct spaces with a dual space.

Example 1. If, in (G2), b and c are points in Z such that $b_\alpha\neq c_\alpha$ for every α, then Σ_b and Σ_c are countably compact and any subspace Y such that either $\Sigma_b\subseteq Y\subseteq Z-\Sigma_c$ or $\Sigma_c\subseteq Y\subseteq Z-\Sigma_b$ has a dual space and $\beta Y=Z$.

This follows from (G2) and the fact that $\beta X\supset Y\supset X$ implies $\beta Y=\beta X$.

Example 2. If, in (G2), b is a point such that each coordinate b_α is not a cluster point of a sequence of X_α, then Σ^b is countably compact and any subspace Y such that either $\Sigma_b\subseteq Y\subseteq Z-\Sigma^b$ or $\Sigma^b\subseteq Y\subseteq Z-\Sigma_b$ has a dual space and $\beta Y=Z$.

This is obtained by the same methods as used in the proof of (G2) (or see [6]).

Next we shall notice the following: i) a space Y, in Examples 1 and 2, is always pseudo-compact [5], and ii) an existence of the point b mentioned in Example 2 is shown by the following way: if Y_α is discrete and $X_\alpha=\beta Y_\alpha$, then the point b is given by (b_α) where $b_\alpha\in Y$ for every α.

Example 3. If each X_α is a compactum, then any Σ-product Σ_b, $b\in Z$, has a dual space and Σ_b is a countably compact, non-paracompact, normal space which has a uniform structure by the family of neighborhoods of the diagonal of $\Sigma_b\times\Sigma_b$.

By (C1) and (C2), Σ_b has all properties above except a non-paracompactness. A non-paracompactness follows from the facts that a countably compact space with a complete structure is compact and a paracompact space has a complete structure.

Example 4. Let Σ_b be a Σ-product in Example 3; then $\Sigma_b \times Z$ is a countably compact, non-normal space with a dual space and $\beta(\Sigma_b \times Z) = Z \times Z$.

This follows from the facts that i) (G1), ii) a product of a countably compact normal space with its any compactification is not normal [4, Theorem 1], and iii) a product of a countably compact space with a compact space is always countably compact.

References

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