67. Interior Regularity of Weak Solutions of the Time-Dependent Navier-Stokes Equation

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§ 1. **Introduction.** It is an interesting problem of mathematical physics whether the time-dependent Navier-Stokes equation has a solution or not. To solve this problem, several authors proposed various weak solutions. In particular, E. Hopf[1] proved the existence, but not the uniqueness, of a weak solution which is global in time, whereas Kiselev and Ladyzenskaia[2] showed the local existence and uniqueness of a weak solution of a different type. In this note we show that the latter is actually a regular solution at least in the interior of the domain if the external force is smooth. We first sketch their result. The equation to be solved is

$$\partial u/\partial t-\Delta u+(u\nabla)u=-\nabla p+f, \quad \operatorname{div} u=0 \text{ in } D\subset E^3,$$
$$u|_{t=0}=a, \quad u|_{\partial D}=0 \ (\partial D \text{ is the boundary of } D).$$

Notations. A vector function belongs to C_0^∞ if its components are of class C_0^∞ (i.e. infinitely differentiable with compact support). $\overset{\circ}{K}_1(D)$ is a real Hilbert space obtained from $\overset{\circ}{K}(D)=\{f\,|\,f\in C_0^\infty(D), \operatorname{div} f=0\}$ by completion with the Dirichlet norm. $H_2(D)$ is a real Hilbert space consisting of all twice strongly differentiable vector functions with the norm $\left(\sum\int u_{i\,x_jx_k}^2dx+\sum\int u_{i\,x_m}^2dx+\sum\int u_n^2dx\right)^{1/2}$. $L^2(D)$ is a real Hilbert space of square integrable vector functions with the norm $\|u\|=(u,u)^{1/2}=\left(\sum\int u_i^2dx\right)^{1/2}$.

Assumptions. 1. D is a bounded domain in the three dimensional Euclidean space E^3. 2. The initial value a belongs to $H_2(D)\frown\overset{\circ}{K}_1(D)$. 3. The external force f and its time derivative $\partial f/\partial t$ belong to $L^2(D\times(0,l))$.

Conclusion. There exists a positive constant T such that in the domain $\Omega=D\times(0,T)$ a generalized solution $u(t)=u(x,t)$ exists uniquely with the following properties. 1. $u(t)\in\overset{\circ}{K}_1(D)$ for each t $(0<t<T)$; 2. u, ∇u, $\partial u/\partial t$, $\partial\nabla u/\partial t\in L^2(\Omega)$; 3. $u(t)$, $\nabla u(t)$, $\partial u(t)/\partial t\in L^2(D)$ for each t $(0<t<T)$ and their L^2 norms are bounded in t; 4. $u(t)\to a$ (strongly in $L^2(D)$ as $t\downarrow 0$); 5. For any sufficiently smooth solenoidal vector

1) E. Hopf: Math. Nachrichten, **4**, 213-231 (1950-1951).

2) A. A. Kiselev and O. A. Ladyzenskaia: Izv. Akad. Nauk SSSR, Seriya Mat., **21**, 655-680 (1957).

function $\varphi(x,t)$ with compact support in Ω, the weak equation

$$(1)\qquad \int_0^T\Big(-\Big(\frac{\partial}{\partial t}+\Delta\Big)\varphi,\ u\Big)dt+\int_0^T(\varphi,(u\nabla)u)dt=\int_0^T(\varphi,f)dt$$

holds, where $(f,g)=\int_D f(x)g(x)dx$.

This generalized solution will hereafter be called *K-L* solution.

Our object is to prove the

Theorem.[3] The *K-L* solution is regular (twice continuously differentiable in x and once continuously differentiable in t) in any subdomain of Ω where the external force $f(x,t)$ is Hölder continuous in (x,t).

To prove the theorem we deduce integral representations for $u(x,t)$ and $\nabla u(x,t)$ in the next section, and study the properties of the integral operators involved in § 3. With these aids we prove the theorem in § 4.

Remark 1. The same result holds for the *K-L* solution in the two-dimensional space.

Remark 2. We have not been able to prove the regularity of Hopf's weak solution.

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§ 2. **Integral representations.** In this section we derive integral representations of u and ∇u. Setting $f'=f-(u\nabla)u$ and taking $\varphi=\text{rot}\,\text{rot}\,\psi=-\Delta\psi+\text{grad}\,\text{div}\,\psi$ $(\psi\in C_0^\infty(\Omega))$ in (1), we have

$$(2)\qquad \int_0^T\Big(\Big(\Delta+\frac{\partial}{\partial t}\Big)\Delta\psi,\ u\Big)dt=\int_0^T(\text{rot}\,\text{rot}\,\psi,\ f')dt.$$

We denote by $\Phi^*(x,t;\xi,\tau)$ a fundamental solution of $\Big(\Delta+\frac{\partial}{\partial t}\Big)\Delta v=0$ with its singularity at the point $(x,t)=(\xi,\tau)$. The explicit form of Φ^* is

$$\Phi^*(x,t;\xi,\tau)=\begin{cases}\dfrac{1}{4\pi^{3/2}|\xi-x|(\tau-t)^{1/2}}\displaystyle\int_0^{|\xi-x|}\exp\Big(-\frac{\rho^2}{4(\tau-t)}\Big)d\rho, & t<\tau,\\ 0, & t>\tau.\end{cases}$$

That Φ^* is a fundamental solution is evident from the fact that

$$-\Delta\Phi^*(x,t;\xi,\tau)=(4\pi(\tau-t))^{-3/2}\exp\Big(-\frac{|\xi-x|^2}{4(\tau-t)}\Big)=E^*(x,t;\xi,\tau)$$

is a fundamental solution of the adjoint heat equation. By the way, we have symbolically

$$(3)\qquad (\Delta+\partial/\partial t)\Delta\Phi^*(x,t;\xi,\tau)=-(\Delta+\partial/\partial t)E^*(x,t;\xi,\tau)=\delta(x-\xi,t-\tau).$$

Next we fix a truncating function $\eta(x,t;\xi,\tau)=\eta(x-\xi,t-\tau)$ such

3) Then there exists a continuous function $p(x,t)$ with continuous space derivatives, and together with this p, $u(x,t)$ is a genuine solution of the N-S eq. in the said subdomain (see Hopf[1]).

that $\eta(x, t)$ is an even C_0^∞-function, equal to one if $|x|<\delta/2$ and $|t|<\delta/2$, and equal to zero if either $|x|>\delta$ or $|t|>\delta$. We take

$$\psi(x, t)=\int_t^T \int_D (\Phi^*\eta)(x, t; \xi, \tau)\chi(\xi, \tau)d\xi\, d\tau, \qquad \chi\in C_0^\infty(\Omega_\delta),$$

as a testing vector function in (2), where $\Omega_\delta=\{(x, t)\,|\,(x, t)\in\Omega,\ \mathrm{dist}\,((x, t), \partial\Omega)>\delta\}$, i.e. Ω_δ is obtained by taking off the boundary strip of width δ from Ω. By virtue of (3) we have

$$(\Delta+\partial/\partial t)\Delta\psi(x, t)=\chi(x, t)-\int_t^T \int_D B^*(x, t; \xi,\tau)\chi(\xi,\tau)d\xi\, d\tau\equiv\chi(x, t)-B^*\chi(x,t),$$

and setting $T^*=(T^*_{ij})$, $T^*_{ij}=T^*_{ji}=-\Delta\Phi^*\delta_{ij}+\partial^2\Phi^*/\partial x_i\partial x_j$,

$$\begin{aligned}(\mathrm{rot\ rot}\,\psi(x, t))_i&=\int_t^T \int_D \Big[-\Delta(\Phi^*\eta)(x, t; \xi, \tau)\delta_{ij}+\frac{\partial^2}{\partial x_i\partial x_j}(\Phi^*\eta)(x, t; \xi, \tau)\Big]\\ &\qquad\times\chi_j(\xi, \tau)d\xi\, d\tau\\ &=\int_t^T \int_D (T^*_{ij}\eta)(x, t; \xi, \tau)\chi_j(\xi, \tau)d\xi\, d\tau+\int_t^T \int_D C^*_{ij}(x, t; \xi, \tau)\\ &\qquad\times\chi_j(\xi, \tau)d\xi\, d\tau\\ &\equiv(T^*_\eta\chi+C^*\chi)_i\end{aligned}$$

where B^*, C^*_{ij} are C_0^∞-functions vanishing identically near $x=\xi$, $t=\tau$, and $C^*_{ij}=C^*_{ji}$. Inserting these relations into (2), we get

$$(4)\qquad \int_0^T (\chi-B^*\chi, u)dt=\int_0^T (T^*_\eta\chi+C^*\chi, f')dt.$$

Carrying out the change of the order of integration, which is justified readily in virtue of the integrability of $f'=f-(u\nabla)u$ and $\chi\in C_0^\infty(\Omega_\delta)$, we have

$$(5)\qquad \int_0^T (\chi, u-Bu)dt=\int_0^T (\chi, (T_\eta+C)f')dt,$$

where

$$(6)\qquad T_\eta f'(x, t)=\int_0^t \int_D (T\eta)(x, t; \xi, \tau)f'(\xi, \tau)d\xi\, d\tau,$$

and $T=(T_{ij})$ is obtained by interchanging (x, t) and (ξ, τ) in $T^*=(T^*_{ij})$. B, C are obtained from B^*, C^* in a similar way. Since χ is arbitrary in (5), we arrive at an integral representation

$$(7)\qquad u(x, t)=T_\eta f'(x, t)+Bu(x, t)+Cf'(x, t), \quad \text{for } (x, t)\in\Omega_\delta.$$

We next derive an integral representation for $\partial u/\partial x_m$, $m=1, 2, 3$. To this end we replace χ by $\partial\chi/\partial x_m$ in (4), and proceed as above, obtaining

$$(8)\qquad \frac{\partial u}{\partial x_m}(x, t)=S^m_\eta f'(x, t)+E^m u(x, t)+F^m f'(x, t), \quad \text{for } (x, t)\in\Omega_\delta,$$

where S^m_η is an integral operator with the kernel $\partial T_{ij}/\partial x_m\eta$, namely

$$(S^m_\eta f'(x, t))_i=\int_0^t \int_D \frac{\partial T_{ij}}{\partial x_m}\eta(x, t; \xi, \tau)f'_j(\xi, \tau)d\xi\, d\tau,$$

and E^m, F^m are integral operators with kernels $E^m(x,t;\xi,\tau)$, $F^m(x,t;\xi,\tau)$ which belong to C_0^∞ vanishing near $x=\xi$, $t=\tau$. As a result we obtain

Lemma 1. For *K-L* solution $u(x,t)$ we have the integral representations (7) and (8).

§3. **Properties of the integral operators.** We now introduce the space $L^p(D)$ with the ordinary L^p norm which we denote by $\|\ \|_p$.

Lemma 2. Let $f=T_\eta g$, $h=S_\eta^m g$ in $\Omega=D\times(0,T)$.

1) Let $g(t)=g(x,t)$ belong to $L^q(D)$ for each t $(0<t<T)$ with its L^q norm bounded. Then for each t $(0<t<T)$ and for any fixed r such that $r^{-1}>q^{-1}-2/3$, $f(t)=f(x,t)$ belongs to $L^r(D)$ and its L^r norm is bounded in t. For each t $(0<t<T)$ and for any fixed s such that $s^{-1}>q^{-1}-1/3$, $h(t)=h(x,t)$ belongs to $L^s(D)$ and its L^s norm is bounded in t.

2) Let $g(x,t)$ be bounded in Ω. Then $f(x,t)$ and $h(x,t)$ are Hölder continuous in Ω_δ with respect to (x,t).

3) Let $g(x,t)$ be Hölder continuous in Ω with respect to (x,t). Then $f(x,t)$ is twice differentiable with respect to x and once differentiable with respect to t in Ω_δ, and these derivatives are continuous in (x,t).

The proof of this lemma is straightforward but cumbersome, so we omit it.

§4. **Proof of Theorem.** Without loss of generality we can assume that the external force $f(x,t)$ is Hölder continuous in Ω. Otherwise we need only to replace Ω by a suitable subset of it. Our proof depends on iterative use of Lemma 2. We first show that u and ∇u are bounded in $\Omega_{4\delta}$. By virtue of section 1, conclusion 3 and the assumption on $f(x,t)$, we see that $\|u(t)\|\le M$, $\|(u\nabla)u(t)\|_1\le M$, and $|f(x,t)|\le M$. Therefore, the second and third terms of the second members of (7) and (8) are bounded in Ω_δ. By a lemma of Sobolev type we know that $\|u(t)\|_6\le c\|\nabla u(t)\|\le M$. Hence $(u\nabla)u\in L^q(D)$ with $q^{-1}=1/2+1/6$ or $q=3/2$. Thus $f'=f-(u\nabla)u\in L^{3/2}(D)$, and its norm is bounded in t. Then the application of Lemma 2, 1) to (7) yields the result that $u\in L^r(D_\delta)$ $(1\le r<\infty)$. From this information we see that $(u\nabla)u\in L^{2-\varepsilon}(D_\delta)$, hence $f'\in L^{2-\varepsilon}(D_\delta)$. A second application of Lemma 2, 1) to (7) gives that $u\in L^r(D_{2\delta})$ if $r^{-1}>1/2-2/3=-1/6$. Hence u is bounded in $D_{2\delta}$ and its bound is bounded in t. Thus we can say that u is bounded in $\Omega_{2\delta}$. This result ensures that $(u\nabla)u\in L^2(D_{2\delta})$ and its norm is bounded in t $(2\delta<t<T-2\delta)$. By a third application of Lemma 2, 1) to (8), $\nabla u\in L^{6-\varepsilon}(D_{3\delta})$ and its norm is bounded in t $(3\delta<t<T-3\delta)$. Hence $(u\nabla)u\in L^{6-\varepsilon}(D_{3\delta})$ and its norm is bounded in t. Again we apply Lemma 2, 1) to (8), and see that ∇u is bounded in $D_{4\delta}$ and its bound is bounded in t. Thus we arrive at the conclusion that u and ∇u are bounded in $\Omega_{4\delta}$.

Next we apply Lemma 2, 2) to (7) and (8), and we see that u and ∇u are Hölder continuous in $\Omega_{5\delta}$, noticing this time that the second and third terms of the second members in (7) and (8) are in $C^\infty(\Omega_{5\delta})$. Finally applying Lemma 2, 3), we see that u is twice continuously differentiable in x, and once continuously differentiable in t in the domain $\Omega_{6\delta}$. This completes the proof.