66. The Space of Bounded Solutions of the Equation $\Delta u = pu$ on a Riemann Surface

By Mitsuru NAKAI
Mathematical Institute, Nagoya University

(Comm. by K. KUNUGI, M.J.A., May 19, 1960)

Throughout this note we denote by R a Riemann surface. Suppose that p is a collection $\{p(z)\}$ of non-negative continuously differentiable functions $p(z)$ of local parameters $z=x+iy$ such that for any two members $p(z)$ and $p(z')$ in p there holds the relation

$$p(z')=p(z)|dz/dz'|^2.$$

We say that such a p is a *density* on R. We consider the partial differential equation of elliptic type

$$\Delta u(z)=p(z)u(z), \tag{1}$$

which is invariantly defined on R. We denote by $B_p(R)$ the totality of real-valued bounded solutions of this equation (1) on R. Here a solution of (1) is always assumed to be twice continuously differentiable. Then $B_p(R)$ is a Banach space with the uniform norm

$$||u||=\sup_R |u|.$$

We are interested in the comparison problem of Banach space structures of $B_p(R)$ for different choices of densities p. It is remarked, as Ozawa proved in [3], that if R is of parabolic type, then $B_0(R)$ is the real number field and $B_p(R)$ consists of only zero unless $p\equiv 0$. Hence we may exclude this trivial case as far as we are concerned with spaces $B_p(R)$. So we assume that R is of hyperbolic type throughout this note unless the contrary is stated. Concerning this comparison problem Royden [4] proved that if there exists a positive constant a such that

$$a^{-1}p \leq q \leq ap$$

holds on R except a compact subset of R, then Banach spaces B_p and B_q are isomorphic. In this note we give a different criterion for B_p and B_q to be isomorphic and state an application of this to removable singularities of bounded solutions of (1).

Theorem 1. *If two densities p and q on R satisfy the condition*

$$\iint_R |p(z)-q(z)|\,dxdy<\infty, \tag{2}$$

then Banach spaces $B_p(R)$ and $B_q(R)$ are isomorphic.

Proof.[1] Let $\{R_n\}$ be an exhaustion of R, i.e. R_n is a subdomain of R whose closure is compact and whose relative boundary ∂R_n consists of a finite number of closed analytic Jordan curves and moreover

1) For elementary knowledge concerning the equation $\Delta u=pu$ on a Riemann surface, refer to Myrberg [1, 2] and also to Royden [4, section 1].

$\{R_n\}$ satisfies

$$\overline{R}_n \subset R_{n+1}; \qquad R=\bigcup_{n=1}^{\infty} R_n.$$

For real-valued bounded continuous function f defined on R, we define transforms Tf and tf as follows:

$$(Tf)(z_0)=f(z_0)+(2\pi)^{-1}\iint_R (p(z)-q(z))g_q(z, z_0)f(z)\, dxdy$$

and

$$(tf)(z_0)=f(z_0)+(2\pi)^{-1}\iint_R (q(z)-p(z))g_p(z, z_0)f(z)\, dxdy,$$

where $g_p(z, z_0)$ and $g_q(z, z_0)$ are Green's functions of R with poles z_0 associated with the equations $\Delta u=pu$ and $\Delta u=qu$ respectively. These are well defined in virtue of the condition (2). We also define auxiliary transforms T_nf and t_nf of real-valued bounded continuous function f defined on R_n as follows:

$$(T_nf)(z_0)=f(z_0)+(2\pi)^{-1}\iint_{R_n} (p(z)-q(z))g_q^{(n)}(z, z_0)f(z)\, dxdy$$

and

$$(t_nf)(z_0)=f(z_0)+(2\pi)^{-1}\iint_{R_n} (q(z)-p(z))g_p^{(n)}(z, z_0)f(z)\, dxdy,$$

where $g_p^{(n)}(z, z_0)$ and $g_q^{(n)}(z, z_0)$ are Green's functions of R_n with poles z_0 associated with the equations $\Delta u=pu$ and $\Delta u=qu$ respectively.

If g is continuous on $\overline{R}_n$ and is a solution of $\Delta u=pu$ (or $\Delta u=qu$) on R_n, then T_ng (or t_ng) is continuous on $\overline{R}_n$ and satisfies the equation $\Delta u=qu$ (or $\Delta u=pu$) on R_n and also

$$\|T_ng\|_{R_n}=\|g\|_{R_n} \quad (\text{or } \|t_ng\|_{R_n}=\|g\|_{R_n}). \tag{3}$$

To verify this, we take a small circle U_η with radius η around z_0 and a subdomain G^ε of R_n such that $\overline{G}^\varepsilon\subset R_n$ and $G^\varepsilon\nearrow R_n$ as $\varepsilon\searrow 0$ and ∂G^ε consists of the same number as ∂R_n of analytic closed Jordan curves and put $G^\varepsilon_\eta=G^\varepsilon-U_\eta$. Let h (or h_ε) be the solution of Dirichlet problem with respect to the equation $\Delta u=qu$ and the domain R_n (or G^ε) with the boundary value g on ∂R_n (or ∂G^ε). Using Green's formula we have

$$\begin{aligned}
&\iint_{G^\varepsilon_\eta} (p(z)-q(z))g_q^{(\varepsilon)}(z, z_0)g(z)\, dxdy\\
&=\iint_{G^\varepsilon_\eta} (g_q^{(\varepsilon)}(z, z_0)d({}^*dg(z))-g(z)d({}^*dg_q^{(\varepsilon)}(z, z_0)))\\
&=\int_{\partial G^\varepsilon_\eta} (g_q^{(\varepsilon)}(z, z_0){}^*dg(z)-g(z){}^*dg_q^{(\varepsilon)}(z, z_0))\\
&=-\int_{\partial U_\eta} g_q^{(\varepsilon)}(z, z_0){}^*dg(z)-\int_{\partial G^\varepsilon} g(z){}^*dg_q^{(\varepsilon)}(z, z_0)+\int_{\partial U_\eta} g(z){}^*dg_q^{(\varepsilon)}(z, z_0),
\end{aligned} \tag{4}$$

where $g_q^{(\varepsilon)}(z, z_0)$ is the Green's function of G^ε with pole z_0 associated with the equation $\Delta u = qu$. It is easy to see that

$$(5) \qquad \int_{\partial U_\eta} g_q^{(\varepsilon)}(z, z_0)^* dg(z) = O(\eta)$$

and

$$(6) \qquad -\int_{\partial G^\varepsilon} g(z)^* dg_q^{(\varepsilon)}(z, z_0) = -\int_{\partial G^\varepsilon} h_\varepsilon(z)^* dg_q^{(\varepsilon)}(z, z_0) = 2\pi h_\varepsilon(z_0)$$

and

$$(7) \qquad \int_{\partial U_\eta} g(z)^* dg_q^{(\varepsilon)}(z, z_0) = -2\pi g(z_0) + O(\eta).$$

From (4), (5), (6) and (7), we get

$$h_\varepsilon(z_0) = g(z_0) + (2\pi)^{-1} \iint_{G_\eta^\varepsilon} (p(z) - q(z)) g_q^{(\varepsilon)}(z, z_0) g(z)\, dxdy + O(\eta).$$

Hence making $\eta \searrow 0$, we see that

$$(8) \qquad h_\varepsilon(z_0) = g(z_0) + (2\pi)^{-1} \iint_{G^\varepsilon} (p(z) - q(z)) g_q^{(\varepsilon)}(z, z_0) g(z)\, dxdy.$$

As $g - h$ is uniformly continuous on $\overline{R}_n$ and vanishes on ∂R_n, so we have $\lim_{\varepsilon \downarrow 0} \sup_{\partial G^\varepsilon} |g - h| = 0$ or $\lim_{\varepsilon \downarrow 0} \sup_{\partial G^\varepsilon} |h_\varepsilon - h| = 0$. From this, using maximum principle, we see that $\lim_{\varepsilon \downarrow 0} \|h_\varepsilon - h\|_{G^\varepsilon} = 0$. In particular

$$(9) \qquad \lim_{\varepsilon \downarrow 0} h_\varepsilon(z_0) = h(z_0).$$

On the other hand, $g_q^{(\varepsilon)}(z, z_0) \nearrow g_q^{(n)}(z, z_0)$ as $\varepsilon \searrow 0$ and

$$|p(z) - q(z)|\, g_q^{(\varepsilon)}(z, z_0)\, |g(z)| \le |p(z) - q(z)|\, g_q^{(n)}(z, z_0)\, |g(z)|$$

and the latter is integrable on R_n. Thus, by Lebesgue's convergence theorem,

$$(10) \qquad \begin{aligned} &\lim_{\varepsilon \downarrow 0} \iint_{G^\varepsilon} (p(z) - q(z)) g_q^{(\varepsilon)}(z, z_0) g(z)\, dxdy \\ &= \iint_{R_n} (p(z) - q(z)) g_q^{(n)}(z, z_0) g(z)\, dxdy. \end{aligned}$$

From (8), (9) and (10), we see that $h(z_0) = (T_n g)(z_0)$. This proves our first assertion. The equality (3) is now a direct consequence of the maximum principle. Similarly, the assertion concerning t_n is verified.

From the above, we easily see that

$$(11) \qquad t_n(T_n g) = g \qquad (\text{or } T_n(t_n g) = g)$$

for any g continuous on $\overline{R}_n$ and satisfying $\Delta u = pu$ (or $\Delta u = qu$) on R_n.

On the other hand, if a uniformly bounded sequence $\{f_n\}$ of real-valued continuous functions f_n defined on R_n converges to a function f defined on R uniformly on each compact subset of R, then for each point z_0 in R

(12) $$(Tf)(z_0)=\lim_n(T_nf)(z_0).$$

In fact, let K be an arbitrary compact subset of R and $|f_n|\leq M$ for all n. As $g_q(z,z_0)-g_q^{(n)}(z,z_0)\searrow 0$ uniformly on each compact subset of R, so we get

$$a_n(z_0)=\left|\iint_R(p(z)-q(z))g_q(z,z_0)f(z)dxdy-\iint_{R_n}(p(z)-q(z))g_q^{(n)}(z,z_0)\right.$$
$$\left.\times f_n(z)\,dxdy\right|\leq(\|f-f_n\|_K+\|g_q-g_q^{(n)}\|_K M)\iint_K|p(z)-q(z)|\,g_q(z,z_0)\,dxdy$$
$$+2M\iint_{R-K}|p(z)-q(z)|\,g_q(z,z_0)\,dxdy.$$

From this we have

$$\overline{\lim}_n\, a_n(z_0)\leq 2M\iint_{R-K}|p(z)-q(z)|\,g_q(z,z_0)\,dxdy.$$

In virtue of the condition (1), letting $K\nearrow R$, we see that

(13) $$\lim_n a_n(z_0)=0.$$

Then the assertion (12) follows from (13) and from the inequality

$$|(Tf)(z_0)-(T_nf_n)(z_0)|\leq|f(z_0)-f_n(z_0)|+a_n(z_0).$$

Now take a function u in $B_p(R)$ (or $B_q(R)$). From (3), the sequence $\{T_nu\}$ (or $\{t_nu\}$) is bounded by $\|u\|$ in the absolute value. Hence by (12) we see that

(14) $$Tu=\lim_nT_nu \qquad (\text{or } tu=\lim_nt_nu)$$

and

(15) $$\|T_nu\|\leq\|Tu\|\leq\|u\| \qquad (\text{or } \|t_nu\|\leq\|tu\|\leq\|u\|),$$

where the convergence is uniform on each compact subset of R by the Harnack type inequality. Hence Tu (or tu) belongs to $B_p(R)$ (or $B_q(R)$). In virtue of (14) and (15), we may apply (12) to (11) with $g=u$ and then we get

$$t(Tu)=u \qquad (\text{or } T(tu)=u).$$

This shows that T (or t) is a one to one mapping of $B_p(R)$ (or $B_q(R)$) onto $B_q(R)$ (or $B_p(R)$) and $T=t^{-1}$. As T and t do not increase norm, so T and t are isometric. Thus Banach spaces $B_p(R)$ and $B_q(R)$ are isomorphic. This completes the proof of Theorem 1.

Assume that a part Γ of the ideal boundary of R can be realized in a larger surface R' as a relative boundary consisting of a finite number of analytic closed Jordan curves and p is the restriction on R of a density on R'. In this case, we denote by $B_p^\Gamma(R)$ the subspace of $B_p(R)$ consisting of every function in $B_p(R)$ which vanishes continuously on Γ. With an obvious modification of the proof of Theorem 1, we can prove the following

Theorem 1′. *Under the assumption that*

(2′) $$\iint_R|p(z)-q(z)|\,dxdy<\infty,$$

Banach spaces $B_p^\Gamma(R)$ *and* $B_q^\Gamma(R)$ *are isomorphic.*

Remark. From the proof we see at once that the assumption (2) in our Theorem 1 (or 1′) can be replaced by the following weaker one:

$$\iint_R |p(z)-q(z)|\,(g_p(z, z_0)+g_q(z, z_1))\, dxdy<\infty \tag{16}$$

for some points z_0 and z_1 in R. In the case $q\equiv 0$, (13) is equivalent to the following

$$\iint_R p(z)g_0(z, z_0)\, dxdy<\infty \tag{17}$$

for some point z_0 in R. Hence in particular we conclude that under the assumption (17), Banach spaces $HB=B_0$ and B_p are isomorphic. It is an open question whether or not (14) is also a necessary condition for HB and B_p to be isomorphic.

Let p be a density on R. A compact subset E in R is said to be B_p-*removable* if for any subdomain D of R containing E and for any bounded solution u of $\Delta u=pu$ on a component D_E of $D-E$ whose boundary contains the boundary of D can be continued to a solution of $\Delta u=pu$ on D. In this definition, we may assume without loss of generality that $\bar{D}$ is compact and the boundary ∂D of D consists of a finite number of analytic closed Jordan curves. As an application of our comparison theorem, we state

Theorem 2. *For any density* p *on* R, *a compact subset* E *of* R *is* B_p-*removable if and only if the logarithmic capacity of* E *is zero.*[2)]

Proof. First notice that D_E and D are hyperbolic Riemann surfaces. Let p and q be any two densities on R. By maximum principle, it is clear that

$$B_p^{\partial D}(D)=B_q^{\partial D}(D)=\{0\}. \tag{18}$$

As $\bar{D}$ is compact, so we have

$$\iint_{D_E} |p(z)-q(z)|\, dxdy\leq \iint_D |p(z)-q(z)|\, dxdy<\infty. \tag{19}$$

Assume that E is B_p-removable. Then any function u in $B_p^{\partial D}(D_E)$ is the restriction of a solution u' in $B_p^{\partial D}(D)$. Hence by (18), $u\equiv 0$ and so $B_p^{\partial D}(D_E)$ consists of zero only. In virtue of (19), by using Theorem 1′, it holds

$$B_q^{\partial D}(D_E)\cong B_p^{\partial D}(D_E)=\{0\}.$$

Hence $B_q^{\partial D}(D_E)$ consists of zero only.

Let v be an arbitrary element in $B_q(D_E)$. We may assume without loss of generality that v is continuous on $\partial D\cup D_E$. Let v' be continuous on $\bar{D}$ and $v'=v$ on ∂D and $\Delta v'=qv'$ in D. Putting $v''=v'-v$,

2) The "if part" of this theorem was proved by Myrberg [2]. Professor M. Ozawa kindly informed me that he has also obtained the same result as our Theorem 2.

we see that v'' is in $B_q^{\partial D}(D_E)$ and hence $v'' \equiv 0$ or $v' \equiv v$ on D_E. Thus E is B_q-removable.

Hence we have proved that for any two densities p and q on R, E is B_p-removable if and only if E is B_q-removable. In particular, taking $q \equiv 0$, and noticing that B_0-removable set is nothing but a set of logarithmic capacity zero, we get the assertion of our theorem.

References

[1] L. Myrberg: Über die Integration der Differentialgleichung $\Delta u = c(P)u$ auf offenen Riemannschen Flächen, Math. Scand., **2**, 142–152 (1954).

[2] L. Myrberg: Über die Existenz der Greenschen Funktion der Gleichung $\Delta u = c(P)u$ auf Riemannschen Flächen, Ann. Acad. Sci. Fenn., A. I., **170** (1954).

[3] M. Ozawa: Classification of Riemann surfaces, Kôdai Math. Sem. Rep., **3**, 63–76 (1952).

[4] H. L. Royden: The equation $\Delta u = Pu$, and the classification of open Riemann surfaces, Ann. Acad. Sci. Fenn., A. I., **271** (1959).