65. *Adjoint Space and Dual Space*

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1. Introduction. Concerning the relation between a locally convex space E and its dual space E' (the totality of continuous linear functionals), the conditions of the semi-reflexivity, the reflexivity, etc. are well known, under the foundation of Mackey's theorem (for ex. [1]). On the other hand, as a space of linear functionals on E, there is a method to consider its *adjoint space* $\overline{E}$ (the totality of linear functionals which are bounded on each bounded set in E) and the condition of reflexivity is known [2].

In this paper, we consider the relation between the adjoint space $\overline{E}$ and the dual space E' of a locally convex separative space E, especially, we give a theorem of Mackey's type with respect to the adjoint space (THEOREM 1).

For a locally convex separative topology T of a linear space E, its adjoint space and its dual space are denoted by $\overline{(E;T)}$ and $(E;T)'$, respectively. Two locally convex topologies T_1 and T_2 of E are said to be *equivalent with respect to bounded set*, when the concepts of the boundedness under T_1 and T_2 are identical, and this equivalence is denoted by $T_1 \overset{b}{\sim} T_2$.

2. A theorem of Mackey's type. Let (E, F) be a separative dual system of two linear spaces. The necessary and sufficient condition that a locally convex separative topology T on E is compatible with the dual system (E, F), that is, $(E;T)'=F$, is that T is stronger than the topology $\sigma(E, F)$ and weaker than the Mackey's topology $\tau(E, F)$ (theorem of Mackey). The following theorem is one of this type concerning the adjoint space.

THEOREM 1. *Let (E, F) be a separative dual system of two linear spaces. The necessary and sufficient condition that F is the adjoint space of E with a locally convex separative topology T on E, or $\overline{(E;T)}=F$ symbolically, is that:*

1. $T \overset{b}{\sim} \sigma(E, F)$
2. *the Mackey's topology $\tau(E, F)$ is bornologic.*

PROOF. Necessity. We denote by $\mathfrak{A}$, the totality of bounded sets under the topology T, and consider a topology T_0 on E defined by all disks which absorb each bounded set under the topology T. T_0 is called $\mathfrak{A}$-topology.*) This topology T_0 is bornologic and the totality of bounded sets under T_0 is also $\mathfrak{A}$, or $T_0 \overset{b}{\sim} T$. Then $(E; T_0)' = \overline{(E; T_0)} = F$ under our assumption. Therefore, $\mathfrak{A}$ is the totality of bounded sets under the topology $\sigma(E, F)$, or $T \overset{b}{\sim} \sigma(E, F)$ and T_0 coincides with the topology $\tau(E, F)$, or $\tau(E, F)$ is bornologic.

Sufficiency. The adjoint space of E with the topology $\tau(E, F)$ coincides with F on account of our assumption 2. Then $\overline{(E; T)} = F$ because of the condition 1.

REMARK 1. When the adjoint space of E with a topology T is F, the topology T is weaker than the topology $\tau(E, F)$. But T is not necessarily comparable with the topology $\sigma(E, F)$. In other words, the dual space of E with the topology T is not necessarily the space F.

REMARK 2. For a separative dual system (E, F), there exists, always, a topology T on E by which the dual space is F. However, there exists not necessarily a topology of E by which the adjoint space is F. For example, when the topology of E is "tonnelé" and is not bornologic, there is no topology T on E by which the adjoint space is E' for the dual system (E, E'), because the topology "tonnelé" is not else than the topology $\tau(E, E')$. Whence, the adjoint space is not E'.

3. Semi-reflexivity with respect to adjoint space. Let E be a locally convex separative space. We denote by T its topology and by $\mathfrak{A}$ the totality of bounded sets under the topology T. On the adjoint space $\overline{E}$, the topology of $\mathfrak{A}$-convergence is denoted by $\beta(\overline{E}, E)$ and the totality of bounded sets under $\beta(\overline{E}, E)$ is denoted by $\overline{\mathfrak{A}}$. In the same way, from the space $\overline{E}$ with the topology $\beta(\overline{E}, E)$, its adjoint space $\overline{\overline{E}}$ and the topology $\beta(\overline{\overline{E}}, \overline{E})$ of $\overline{\mathfrak{A}}$-convergence are defined. The topology on E induced by $\beta(\overline{\overline{E}}, \overline{E})$ is the $\mathfrak{A}$-topology used in the above paragraph, and it coincides with the initial topology T, if and only if T is bornologic. When $\overline{\overline{E}}$ coincides with E algebraically, that is, $\overline{(\overline{E}; \beta(\overline{E}, E))} = E$, E is said to be *adjoint semi-reflexive*.

THEOREM 2. *The necessary and sufficient condition that a locally convex separative space E is adjoint semi-reflexive, is that:*

1. *every bounded disk in E is $\sigma(E, \overline{E})$ relatively compact,*

2. *the topology $\beta(\overline{E}, E)$ is bornologic.*

PROOF. When we apply THEOREM 1 to the dual system $(\overline{E}, E)$, the condition of adjoint semi-reflexivity is that:

1′. $\beta(\overline{E}, E \overset{b}{\sim} \sigma(\overline{E}, E)$,

*) Let $\overline{\mathfrak{A}}$ be the totality of bounded sets under the topology of $\mathfrak{A}$-convergence on $\overline{E} = \overline{(E; T)}$, then $\mathfrak{A}$-topology is not else than the topology of $\overline{\mathfrak{A}}$-convergence on E with respect to the dual system $(E, \overline{E})$.

2'. $\tau(\overline{E}, E)$ is bornologic.

Now, we assume the conditions 1', 2'. Generally, the topology $\beta(\overline{E}, E)$ is stronger than the topology $\tau(\overline{E}, E)$. On account of the condition 1' the topology $\beta(\overline{E}, E)$ is equivalent to the topology $\tau(\overline{E}, E)$ with respect to bounded set. Then $\beta(\overline{E}, E)$ coincides with $\tau(\overline{E}, E)$ because of the assumption 2'. Therefore the topology $\beta(\overline{E}, E)$ is bornologic and every bounded disk in E is $\sigma(E, \overline{E})$ relatively compact.

Next, we assume the conditions 1, 2. On account of the condition 1, the topology $\tau(\overline{E}, E)$ coincides with the topology $\beta(\overline{E}, E)$. Then the topology $\tau(\overline{E}, E)$ is bornologic and $\beta(\overline{E}, E) \overset{b}{\sim} \sigma(\overline{E}, E)$.

COROLLARY 1. *If a locally convex separative space E is adjoint semi-reflexive, then E is semi-reflexive in the ordinary sense.*

Because the topology $\sigma(E, \overline{E})$ is stronger than the topology $\sigma(E, E')$.

COROLLARY 2. *If a locally convex separative space E is adjoint semi-reflexive, then its adjoint space $\overline{E}$ with the topology $\beta(\overline{E}, E)$ is adjoint reflexive.*

Because, the topologies $\beta(\overline{E}, E)$ and $\beta(E, \overline{E})$ are both bornologic.

COROLLARY 3. *A locally convex separative space E is adjoint reflexive when and only when the following conditions are satisfied:*

1. *the initial topology on E is bornologic,*
2. *the topology $\beta(\overline{E}, E)$ on $\overline{E}$ is bornologic,*
3. *every bounded disk in E is $\sigma(E, \overline{E})$ relatively compact* [2].

COROLLARY 4. *If a locally convex separative space E is adjoint reflexive, then E is reflexive in the ordinary sense.*

REMARK 1. If A' is a $\sigma(E', E)$ relatively compact disk in the dual space E' of a locally convex space E, then A' is bounded under the topology $\beta(E', E)$. But the converse is not necessarily true. However, for a disk $\overline{A}$ in the adjoint space $\overline{E}$, the concepts of $\sigma(\overline{E}, E)$ relative compactness and $\beta(\overline{E}, E)$ boundedness are identical, because the topology $\beta(E, \overline{E})$ coincides with the topology $\tau(E, \overline{E})$ in E.

REMARK 2. The condition 1 in THEOREM 2 means that the dual space of the space $\overline{E}$ with the topology $\beta(\overline{E}, E)$ is the space E. In the same method in the proof of THEOREM 2, we obtain that the adjoint space of the dual space E' with the topology $\beta(E', E)$ is the space E if and only if

1. E is semi-reflexive, 2. $\beta(E', E)$ is bornologic.

4. A condition for $E' = \overline{E}$. We apply THEOREM 1 to the dual system composed of a locally convex separative space E and its dual space E'. Then we obtain:

THEOREM 3. *The adjoint space $\bar{E}$ coincides with the dual space E' for a locally convex separative space E if and only if the topology $\tau(E, E')$ is bornologic.*

That $\bar{E}$ coincides with E' means that the initial topology in E is stronger than the topology $\sigma(E, \bar{E})$. In this direction, we obtain the following condition for $E'=\bar{E}$.

THEOREM 4. *The adjoint space $\bar{E}$ coincides with the dual space E' for a locally convex separative space E if and only if every disk in E satisfying the following conditions is a neighbourhood of 0 in E.*

1. *V absorbs each bounded set in E,*
2. *the quotient space E/F by $F=\bigcap_{\varepsilon>0}\varepsilon V$ is one dimensional.*

PROOF. We assume that $\bar{E}=E'$. Let V be a disk satisfying the conditions 1, 2. If G is an algebraic supplement of F, then G is one dimensional. The projection V_1 of V into G is a disk in G. Let f be a linear functional on G such that $V_1 \subset \{z \in G: |\langle z, f\rangle| \leqq 1\} \subset \lambda V_1$ for some $\lambda > 1$. Every element $x \in E$ is uniquely decomposed into $x=y+z$, $y \in F$, $z \in G$. Then we define a linear functional $\bar{x}$ on E by the relation $\langle x, \bar{x}\rangle = \langle z, f\rangle$. If $x \in V$, $|\langle x, \bar{x}\rangle| = |\langle z, f\rangle| \leqq 1$. Therefore, $\bar{x}$ is bounded linear functional on E because of the condition 1 and so $\bar{x}$ is continuous by our assumption. If $|\langle x, \bar{x}\rangle| \leqq 1$, $x=y+z$, then $y \in F \subset V$, $z \in \lambda V_1 \subset \lambda V$ and so $x \in (\lambda+1)V$. Thus V is a neighbourhood of 0 in E.

Conversely, if $\bar{x}$ is a non zero bounded linear functional on E, then $V=\{x: |\langle x, \bar{x}\rangle| \leqq 1\}$ is a disk satisfying the conditions 1, 2, and so V is a neighbourhood of 0 in E or $\bar{x}$ is continuous.

REMARK. In the same way as the above proof, we obtain the conditions for $E'=E^+$ and $\bar{E}=E^+$ where E^+ is the algebraic dual of E. For an absorbant disk V in E, the former is that if E/F is one dimensional then V is a neighbourhood of 0 and the latter is that if E/F is one dimensional, then V absorbs each bounded set, where $F=\bigcap_{\varepsilon>0}\varepsilon V$ in both cases.

5. An example of $E' \neq \bar{E}$. If the topology $\tau(E, E')$ is not bornologic, then $E' \neq \bar{E}$ (REMARK 2 in the paragraph 2). Here, we give a simple example of such E.

Let R be the segment $[0,1]$ and $\mathfrak{K}$ be the totality at most countable sets in R. Or more generally, let R be an abstract set, and $\mathfrak{K}$ be a family of subsets in R which satisfy the following three conditions:

1. $R \notin \mathfrak{K}$; 2. if $K_i \in \mathfrak{K}$, $i=1,2,\cdots$, then $\bigcup_{i=1}^{\infty} K_i \in \mathfrak{K}$;

3. for each point $t \in R$ there is a set $K \in \mathfrak{K}$ such that $t \in K$.

Let E be the linear space composed of all functions $x=x(t)$ on R

such that each $x=x(t)$ is constant outside of some K: $x(t)=x_K=\text{const.}$ for $t\notin K$. Such K may be called a *supporting set* of x. The topology of E is the topology of simple convergence at each point of R. Let L be a linear functional on E such that $\langle x, L\rangle=x_K$ for $x\in E$.

L is not continuous. If it is continuous, there exists $t_i\in R$, $i=1, 2,\cdots, k$, and a positive number δ such that $x\in E$ $|x(t_i)|\leqq\delta$, $i=1,2,\cdots, k$, implies $|\langle x, L\rangle|\leqq 1$. On the other hand, a function $x=x(t)$ such that $x(t_i)=0$, $i=1,2,\cdots,k$ and $x(t)=2$ otherwise, belongs to E. Then $\langle x, L\rangle=2$. This is a contradiction.

L is bounded on each bounded set. If not so, there is a bounded set A in E such that $|\langle x_n, L\rangle|>n, x_n\in A$, $n=1,2,\cdots$. Let K_n be a supporting set of x_n, and put $K=\bigcup_{n=1}^{\infty}K_n$. Because of $K\neq R$, $x_n(t_0)=\langle x_n, L\rangle$ for $t_0\notin K$, and $|x_n(t_0)|>n$. This contradicts the boundedness of A.

The space is a dense subspace of the space $E_0=\prod_{t\in R}C_t$, where C_t is the scalar field and the topology of E_0 is the product topology of $\{C_t;\, t\in R\}$. Let F be the space $\sum_{t\in R}C_t$, that is, the direct sum of $\{C_t;\, t\in R\}$, then (E, F) is a separative dual system and the topology of E is not else than the topology $\sigma(E, F)$. Now, we show that *each bounded set B in F with the topology $\sigma(F, E)$ is a finite dimensional bounded set*, that is, there exists a finite set $t_i\in R$, $i=1,2,\cdots,k$ such that

$$\sup_{y\in B}|y(t_i)|<M,\ i=1,2,\cdots,k;\ y(t)=0\ \ (t\neq t_i,\ y\in B).$$

If not so, we can select a sequence $t_n\in R$ and $y_n\in B$, $n=1,2,\cdots$ such that: $\quad y_n(t_n)=\eta_n\neq 0;\quad y_n(t_m)=0\quad (m>n).$
We put $k_m=\sup_n|y_n(t_m)|$ $(m=1,2,\cdots)$ which is finite because of the $\sigma(F, E)$ boundedness of B and define an element $x=x(t)\in E$ by $x(t_n)=\xi_n$; $x(t)=0$ otherwise, where $\xi_1=1/|\eta_1|$, $\xi_n=\{n+\xi_1k_1+\cdots+\xi_{n-1}k_{n-1}\}/|\eta_n|$. Then $|\langle x, y_n\rangle|>n$. This is a contradiction.

Thus, the weakly bounded set B in $E'=F$ is equi-continuous, and the topology of E coincides with the topology $\tau(E, F)$. Therefore the topology of E is "tonnelé" but not bornologic and not complete.

6. A disk on which bounded linear functionals are continuous. We characterize a subset A of a locally convex space E such that every linear functional bounded on A is continuous. This characterization is a generalization of the result known for the metrizable case [3]. If a linear functional f is bounded on A, f is also bounded on the disk envelope of A, so we consider only a disk A.

THEOREM 5. *Let A be a disk in a locally convex space E. The following conditions are equivalent each other.*

1. *For any locally convex space E_1 and a linear operator u from E into E_1, if $u(A)$ is bounded in E_1, then u is continuous.*

2. *There is a linear subspace F including A such that the quotient topology in E/F is the strongest convex topology and A is a neighbourhood of 0 in F by the relative topology.*

PROOF. $1 \rightarrow 2$. Let F be a linear subspace of E generated by A, and v be a linear operator from E/F with the quotient topology into E/F with any other locally convex topology. Then $v \circ u$ is a linear operator from E into E/F where u is the canonical operator from E onto E/F, and $v \circ u(A)=0$. Thus $v \circ u$ and so also v is continuous on account of the assumption 1. Therefore the quotient topology of E/F is the strongest convex topology. Hence, F is a topological direct factor of E. Let w be the projection from E onto F. We apply our assumption 1 to the space F with the topology defined by the system of the neighbourhood $\lambda A(\lambda>0)$ of 0 and the linear operator w. Then $w(A)$ is bounded in F. Therefore w is continuous, or there is a neighbourhood U of 0 in E such that $w(U) \subset A$.

$2 \rightarrow 1$. Because of the strongest convexity of E/F, E is decomposed into $F+G$, where G is a topological supplementary of F. For a linear operator u from E into another E_1, the restriction $u|F$ is continuous because it is bounded on A and the restriction $u|G$ is continuous on account of the strongest convexity of $G(\cong E/F)$. Then u is continuous.

COROLLARY. *When the topology of E is $\tau(E, E')$, the conditions 1, 2 in the above Theorem are also equivalent to:*

3. *every linear functional bounded on A is continuous.*

PROOF. $1 \rightarrow 3$. This is obvious.

$3 \rightarrow 1$. In this case, the operator u mentioned in the condition 1, is weakly continuous and then u is continuous because the topology is $\tau(E, E')$.

REMARK. When E/F with the quotient topology is metrizable and infinite dimensional, there is a linear functional which is not continuous. In the metrizable case, the strongest convexity means that the space is finite dimensional, and this is the case considered in [3].

References

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