# 114. On the Cosine Problem

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**1. Introduction.** The main object of the present note is to establish the following theorem, which will answer in the affirmative to the cosine problem proposed by S. Chowla in connexion with a question concerning zeta functions (cf. [1]):

**Theorem 1.** *Let $K$ be an arbitrary positive number. Then there exists a natural number $n_0=n_0(K)$ such that for any $n\geqq n_0$ and any set of $n$ distinct positive integers $m_1, m_2,\cdots, m_n$ we have*

$$\min_{0\leqq x<2\pi}(\cos m_1x+\cos m_2x+\cdots+\cos m_nx)<-K.$$

Here we may take

$$(1)\qquad n_0(K)=\max(2^{48}, [8K^2]^{8[256K^4]}),$$

which is, of course, not the best possible.

As a simple generalization of Theorem 1 we can prove also that, given a real number $K>0$, there is an $n_0=n_0(K)$ such that for any $n\geqq n_0$ and any set of $n$ distinct positive integers $m_1, m_2,\cdots, m_n$ we have

$$\min_{0\leqq x<2\pi}\sum_{j=1}^{n}\cos(m_jx+\omega_j)<-K,$$

where $\omega_1, \omega_2,\cdots, \omega_n$ are arbitrary real numbers, and in particular,

$$\min_{0\leqq x<2\pi}\sum_{j=1}^{n}\sin m_jx<-K,\qquad \max_{0\leqq x<2\pi}\sum_{j=1}^{n}\sin m_jx>K.$$

Thus Theorem 1 is a special case of the following

**Theorem 2.** *Let $G$ be a locally compact connected abelian group. Given a real number $K>0$, we can find an $n_0=n_0(K)$ such that for any $n\geqq n_0$ and any set of $n$ distinct characters $\chi_1(x), \chi_2(x),\cdots, \chi_n(x)$ on $G$ we have*

$$\inf_{x\text{ in }G}\operatorname{Re}\sum_{j=1}^{n}c_j\chi_j(x)<-K,$$

*where $c_1, c_2,\cdots, c_n$ are arbitrary complex numbers with $|c_j|\geqq 1$ $(1\leqq j\leqq n)$.*

For instance, if $\lambda_1, \lambda_2,\cdots, \lambda_n$ are arbitrary distinct positive real numbers, where $n\geqq n_0$, then we have

$$\inf_{x\text{ real}}(\cos\lambda_1x+\cos\lambda_2x+\cdots+\cos\lambda_nx)<-K.$$

**2. Some lemmas.** In order to prove the theorems we appeal to a technique by P. J. Cohen [2] developed in the investigation of a different problem, and so, to avoid ambiguity, we shall here reproduce some of his lemmas given in [2] with a slight modification.

Let $X$ be the interval $[0, 2\pi]$. Let $C$ denote the space of all continuous functions defined on $X$ and $C_0$ be the subset of $C$ consisting of all functions with absolute values not greater than unity. If $\mu$ is a finite measure defined on $X$, we denote by $\|\mu\|$ the norm of $\mu$, i.e.

$$\|\mu\|=\int_X d|\mu|(x).$$

Naturally, to such a measure $\mu$ there corresponds a linear functional $L$ on $C$ with the norm

$$\|L\|=\sup\left|\int_X \phi(x)d\mu(x)\right|=\|\mu\|,$$

where the supremum is taken over all $\phi(x)$ in $C_0$.

In what follows $\mu$ will be supposed to be a finite measure on $X$ such that $\|\mu\|\leqq M$, $M\geqq 1$.

*Lemma* 1. Let $g_j(x)$ $(1\leqq j\leqq r)$ be a set of functions in $C_0$ such that

$$\int g_j(x)d\mu(x)=1 \qquad (1\leqq j\leqq r).$$

Then, if $r>2M^2-1$, we have, for some pair $i<j$,

$$\mathrm{Re}\int g_i(x)\bar{g}_j(x)d|\mu|(x)>\frac{1}{2M}.$$

*Lemma* 2. Let $\phi(x)$ and $g(x)$ be functions in $C_0$ satisfying the following conditions:

$$\int \phi(x)d\mu(x)=A \qquad (A\geqq 1),$$

$$\left|\int g(x)d|\mu|(x)\right|\geqq \alpha \qquad (0<\alpha<1),$$

and

$$\int \phi(x)g(x)d\mu(x)=0.$$

Then

$$\|\mu\|\geqq A+\frac{\alpha^2}{4A}.$$

*Lemma* 3. Let $\phi(x)$ and $g_j(x)$ $(1\leqq j\leqq r)$ be functions in $C_0$ such that

$$\int \phi(x)d\mu(x)=A \qquad (A\geqq 1),$$

$$\int g_j(x)d\mu(x)=1 \qquad (1\leqq j\leqq r),$$

and for all $i<j$,

$$\int \phi(x)g_i(x)\bar{g}_j(x)d\mu(x)=0.$$

Then, if $r>2M^2-1$, we have

$$\|\mu\| \geqq A + \frac{1}{16M^3}.$$

By Lemma 1, for some pair $i<j$ we have

$$\left| \int g_i \bar{g}_j d|\mu| \right| > \frac{1}{2M}.$$

Put, in Lemma 2, $g = g_i \bar{g}_j$ with $\alpha = 1/2M$. Then

$$\|\mu\| \geqq A + \frac{1}{16AM^2} \geqq A + \frac{1}{16M^3},$$

since $A \leqq \|\mu\| \leqq M$.

*Lemma* 4. Under the hypotheses of Lemma 3, there exist constants $a, b_j, c_{ij}$ such that if

$$\psi(x) = a\phi(x) + \sum_j b_j g_j(x) + \sum_{i<j} c_{ij} \phi(x) g_i(x) \bar{g}_j(x),$$

we have $|\psi(x)| \leqq 1$ on $X$ and

$$\int \psi(x) d\mu(x) = A + \frac{1}{16M^3}.$$

Let $V$ denote the linear subspace of $C$ generated by $\phi, g_j$ and $\phi g_i \bar{g}_j$. The measure $\mu$ induces a linear functional $L$ on $V$ with the norm $N$, say. The functional $L$ can be extended to a functional on the whole space $C$ with the same norm $N$, and the new functional is given by a measure satisfying the conditions of Lemma 3. Hence

$$N \geqq A + \frac{1}{16M^3}.$$

From this inequality the result follows at once.

*Lemma* 5. Let $E = \{m_1 > m_2 > \cdots > m_n\}$ be a set of $n$ distinct positive integers. If $r$ and $s$ are natural numbers satisfying

(2) $$r^{3s} \leqq n,$$

then there exist sets $F_1, \cdots, F_{s+1}, G_1, \cdots, G_s$ of positive integers with the following properties:

(a) $F_1 = \{m_1\}$;

(b) for all $k$, $1 \leqq k \leqq s$, $G_k = \{m_{k1} > m_{k2} > \cdots > m_{kr}\}$ is a subset of $E$ and $m + m_{ki} - m_{kj}$ is not contained in $E$ if $m$ is in $F_k$ and $i<j$;

(c) $F_{k+1}$ is the union of $F_k, G_k$ and all integers of the form $m + m_{ki} - m_{kj}$ with $m$ in $F_k$, $i<j$.

We denote by $h(k)$ the smallest integer $h$ such that $m \geqq m_h$ for all $m$ in $F_k$. Assume that the sets $F_1, \cdots, F_k, G_1, \cdots, G_{k-1}$ $(k \geqq 1)$ have been chosen to satisfy the conditions (a), (b) and (c). We now define the set $G_k$. Set $m_{k1} = m_1$. Suppose that $m_{k1}, \cdots, m_{kt}$ $(t \geqq 1)$ have been chosen so as to satisfy (b), where $m_{ki} = m_{j(i)}$ for $i \leqq t$. We then define $m_{k,t+1} = m_{j(t+1)}$, where $j(t+1)$ is the smallest number such that this choice of $m_{k,t+1}$ does not violate (b). The number of choices of $m_{k,t+1} < m_{kt}$ such that

$$m + m_{ki} - m_{k,t+1} \in E$$

for some $m$ in $F_k$ and $m_{ki}$, $i \leqq t$, does not exceed

$$\frac{r(r-1)}{2} h(k).$$

Hence we find that

$$j(t+1) - j(t) \leqq 1 + \frac{r(r-1)}{2} h(k),$$

and

$$h(k+1) = j(r) \leqq r + \frac{r^2(r-1)}{2} h(k) \leqq r^3 h(k),$$

on defining the set $F_{k+1}$ by means of (c). Since $h(1)=1$, it follows that $h(k) \leqq r^{3(k-1)}$. Clearly the sets $F_s$, $G_s$, and hence $F_{s+1}$ can be constructed if $h(s+1) \leqq n$, or

$$r^{3s} \leqq n.$$

That the sets $F_k$ and $G_k$ thus constructed contain only positive integers is obvious.

3. **Proof of Theorem 1.** There is no loss in generality in assuming that $K \geqq 1/2$.*) Suppose now that the theorem were false. Then there would be a real number $K \geqq 1/2$ such that for arbitrarily large $n$ there exist $n$ distinct positive integers $m_1, \cdots, m_n$ for which the inequality

$$\cos m_1 x + \cdots + \cos m_n x \geqq -K$$

holds for all $x$ in $X$. Put

$$f(x) = K + \cos m_1 x + \cdots + \cos m_n x = \frac{1}{2}\Big(2K + \sum_{j=1}^{n} (e^{im_j x} + e^{-im_j x})\Big).$$

Then $f(x) \geqq 0$ throughout on $X$. Now consider the finite, non-negative measure $\mu$ defined on $X$ by

$$d\mu(x) = 2f(x)dx,$$

where $dx$ is $1/2\pi$ times the ordinary Lebesgue measure on $X$. We have

$$\|\mu\| = \int_X d\mu(x) = 2K \geqq 1,$$

and for positive $m$,

$$\int_X e^{imx}\, d\mu(x) = \begin{cases} 1 & \text{if } m = m_j \text{ for some } j, \\ 0 & \text{otherwise.} \end{cases}$$

Without loss of generality we may suppose that $m_1 > \cdots > m_n$. Put $r = [8K^2]$ and $s$ be the largest integer satisfying (2). We construct functions $\phi_k(x)$ $(1 \leqq k \leqq s+1)$, which are to be all in $C_0$, such that each $\phi_k(x)$ is a linear combination of $e^{imx}$ with $m$ in $F_k$ and satisfies

$$\int \phi_k(x)\, d\mu(x) = 1 + \frac{k-1}{128K^3}.$$

Take $\phi_1(x) = e^{im_1 x}$. If $\phi_k(x)$ $(k \geqq 1)$ has already been defined, then

*) For $1/2 \geqq K > 0$ we may take $n_0(K) = 1$.

by Lemmas 4 and 5 with

$$g_j(x)=e^{imx} \qquad (1\leqq j\leqq r),$$

where $m=m_{kj}$ are in $G_k$, we can find a function $\psi(x)=\phi_{k+1}(x)$ in $C_0$ such that $\phi_{k+1}(x)$ is a linear combination of $e^{imx}$ with $m$ in $F_{k+1}$ and

$$\int \phi_{k+1}\, d\mu=1+\frac{k-1}{128K^3}+\frac{1}{128K^3}=1+\frac{k}{128K^3}.$$

Since we must always have

$$\int \phi_k d\mu \leqq ||\mu||=2K,$$

it follows that

$$1+\frac{s}{128K^3}\leqq 2K.$$

If $s=[256K^4]$, this inequality cannot hold, so that necessarily

$$[8K^2]^{3[256K^4]}>n,$$

which is, however, certainly impossible when $n\geqq n_0$, where $n_0=n_0(K)$ is defined in (1). This completes the proof of Theorem 1.

4. **Proof of Theorem 2.** The passage of carrying our proof of Theorem 1 on that of Theorem 2 is substantially as indicated in [2, Lemmas 1′ and 5], and we may omit the details.

## References

[1] S. Chowla: The Riemann zeta and allied functions, Bull. Amer. Math. Soc., **58**, 287–305 (1952).

[2] P. J. Cohen: On a conjecture of Littlewood and idempotent measures, Amer. J. Math., **82**, 191–212 (1960).